Module - 4

4.1. Dynamics

Dynamics deals with the motion of bodies under the action of forces. It has two distinct parts - kinematics and kinetics. Kinematics is the study of motion of bodies without reference to the forces which either cause the motion or are generated as a result of the motion. Kinetics is the study of the relationship between motion and the corresponding forces which cause or accompany the motion.

4.2. Rectilinear transalation

When a particle moves along a straight line, the motion is called rectilinear transalation.

Kinematics of rectilinear translation of a particle is characterised by specifying the displacement, velocity and acceleration of the particle at any given instant.

Displacement

The change of position of a particle with respect to a certain fixed reference point is termed as displacement.

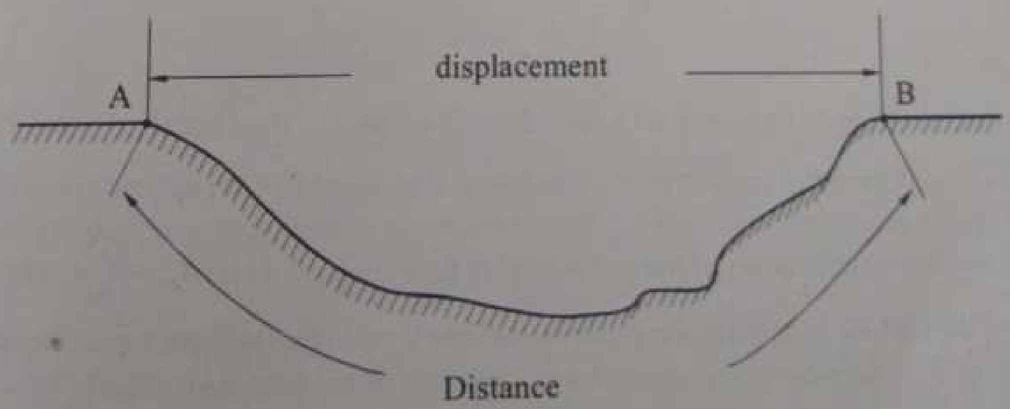


Fig. 4.1

Consider a particle that moves from A to B along a curved path as shown in Fig 4.1, in time t seconds. The length of this curved path in between A and B is called distance covered by the particle in t seconds. The shortest distance between A and B is called displacement of the particle in t seconds. Displacement towards right of a reference point is taken as positive and displacement towards left is taken as negative. Displacement has both magnitude and direction and so it is a vector quantity. Distance is a scalar quantity since it has only magnitude.

Velocity

The rate of change of position of a particle with respect to time is called velocity.

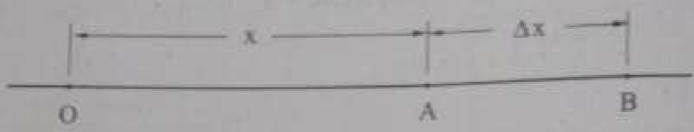


Fig. 4.2

Consider the motion of a particle along a straight line as shown in Fig 4.2. At time t the particle is at A, at a distance x from the reference point O. At time $(t + \Delta t)$ the particle is at B, which is at a distance of $(x + \Delta x)$ from reference point O. The average velocity of the particle over time interval Δt is $V_{av} = \frac{\Delta x}{\Delta t}$. The velocity of the particle at a particular point on the line is called instantaneous velocity of the particle. It is the average velocity in the limit Δt tends to zero. The instantaneous velocity,

$$V = \text{Lim}_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) = \frac{dx}{dt}$$

$$V = \frac{dx}{dt}$$

Acceleration

The rate of change of velocity of a particle with respect to time is called acceleration. If V and $(V + dV)$ are the velocities of a particle at time t and $(t + \Delta t)$ seconds respectively, then the average acceleration of the particle over time interval Δt is, $a_{av} = \frac{\Delta V}{\Delta t}$. The acceleration of the particle at a particular point on the line is called instantaneous acceleration of the particle. It is the average acceleration in the limit Δt tends to zero. The instantaneous acceleration, $a = \text{Lim}_{\Delta t \to 0}\left(\frac{\Delta V}{\Delta t}\right) = \frac{dV}{dt}$

Acceleration , $a = \frac{dV}{dt} = \frac{d}{dt}\left[\frac{dx}{dt}\right] = \frac{d^2x}{dt^2}$

$$a = \frac{d^2x}{dt^2}$$

Again, $a = \frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = \frac{dV}{dx} \times V$

$$a = V \cdot \frac{dV}{dx}$$

When the velocity of a particle decreases, the acceleration will be negative. Negative acceleration is called retardation or deceleration.

4.3 Equations of kinematics

Equations of motion in kinematics relate displacement, velocity, acceleration and time. When a particle moves with constant acceleration, the initial velocity, u, final velocity V, distance s during time t seconds are related by the expressions,

$$V = u + at$$

$$V^2 = u^2 + 2as \text{ and}$$

$$s = ut + \frac{1}{2}at^2$$

For a freely falling bdoy, the acceleration a is the acceleration due to gravity, g. Then the expressions are,

$$v = u + gt$$

$$v^2 = u^2 + 2gh$$

$$h = ut + \frac{1}{2}gt^2$$

When the particle moves upwards, the acceleration a is (−g) and hence, the equations are

$$v = u - gt$$

$$v^2 = u^2 - 2gh$$

$$h = ut - \frac{1}{2}gt^2$$

Example 4.1.

A stone is dropped from the top of a tower, 60 m high. At the same time another stone is thrown upwards from the foot of the tower with a velocity of 30 m/s. When and where the two stones cross each other?

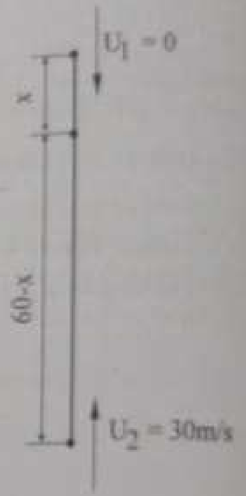
Solution:

Given : height of tower, h = 60 m.

$$u_1 = 0,\ u_2 = 30 \text{ m/s}$$

$$t_1 = t_2 = t$$

Let x be the distance from the top of the tower where the two stones cross each other.



$$x = u_1 t + \frac{1}{2} gt^2$$

$$= 0 + \frac{1}{2} gt^2 \text{ ------- (i)}$$

$$60 - x = u_2 t - \frac{1}{2} gt^2 \text{ ------- (ii)}$$

adding equations (i) and (ii)

$$60 = u_2 \times t$$

$$t = \frac{60}{30} = 2 \text{ s}$$

$$x = u_1 t + \frac{1}{2} g t^2$$

$$= 0 + \frac{1}{2} \times 9.81 \times 2^2$$

$$= 19.62 \text{ m}$$

The two stones will cross each other at a distance of 19.62 m from the top of the tower, after 2 seconds.

Example 4.2.

A stone, dropped into a well, is heared to strike the water after 2 seconds. Find the depth of the well, if the velocity of sound is 340 m/s.

Solution:

Velocity of sound, $V_s = 340$ m/s.

Let h be the depth of well and t_1 is the time taken by the stone to reach the bottom of well and t_2 is the time taken by the sound to reach the top of well.

$$t_1 + t_2 = 2s$$

$$h = \text{velocity of sound} \times \text{time}$$

$$= 340 \times t_2 = 340\,(2 - t_1) \ldots\ldots (i)$$

$$h = ut + \frac{1}{2} g t^2$$

$$h = 0 + \frac{1}{2} \times 9.81 \times t_1^2 \ldots\ldots (ii)$$

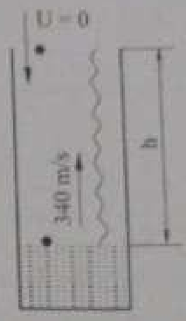


Fig. 4.3

From eqns. (i) and (ii)

$$340\,(2 - t_1) = \frac{1}{2} \times 9.81 \times t_1^2$$

$$69.32\,(2 - t_1) = t_1^2$$

$$t_1^2 + 69.32\,t_1 - 138.64 = 0$$

$$t_1 = 1.95\text{ s}$$

$$t_2 = 2 - t_1 = 2 - 1.95 = 0.05\text{ s}$$

Therefore $h = 340 \times t_2$

$$= 340 \times 0.05$$

$$= 17\text{ m}$$

Velocity-time curve

In the velocity - time diagram, the abscissa represents time of motion and the ordinate represents the velocity.

$$\text{Velocity } V = \frac{dS}{dt}$$

$$dS = V\,dt$$

$$\int dS = \int V\,dt$$

$S = \int V\,dt$. The area under the velocity time curve represents the displacement.

The slope of velocity time curve is $\frac{dV}{dt}$. Since $\frac{dV}{dt} = a$, the slope of the curve represents the acceleration.

Case (i) when the velocity is uniform.

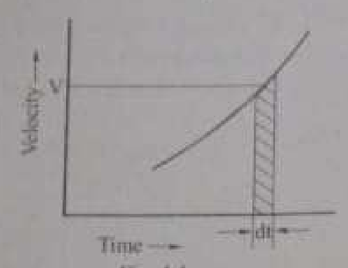


Fig. 4.4

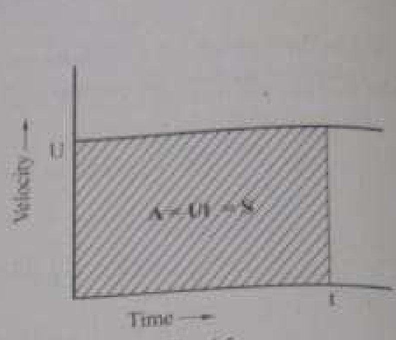


Fig. 4.5

$S = ut + \frac{1}{2} a t^2$, when the velocity is uniform, acceleration $a = 0$. Therefore, $S = u\,t$. Area under the V – t curve, $A = u \times t = S$

Case (ii) When velocity increases linearly from an initial velocity u.

The velocity time diagram is shown in Fig.4.6. Velocity increases from u to V during t seconds.

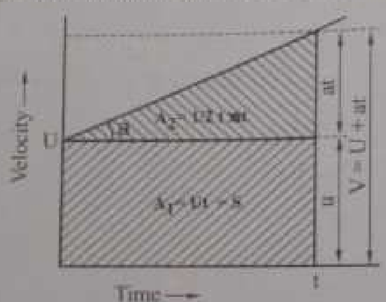


Fig. 4.6

Area under the V – t diagram is $A_1 + A_2$.

$$A = A_1 + A_2 = u\,t + \frac{1}{2}\,t \times a\,t$$

$$= u\,t + \frac{1}{2}\,a\,t^2 = S, \text{ displacement.}$$

Slope of the V-t diagram,

$$\tan\alpha = \frac{at}{t} = a, \text{ acceleration.}$$

Example 4.3.

A train travels between two stopping stations, 7 km apart in 14 minutes. Assuming that its motion is one of uniform acceleration for part of the journey and uniform retardation for the rest, prove that the greatest speed on the journey is 60 km/hr.

Solution

$$S = 7 \text{ km.},$$

$$t = 14 \text{ min.} = (\frac{14}{60})\text{ h}$$

$$S = \frac{1}{2} \times t \times V_{max}$$

$$7 = \frac{1}{2} \times (\frac{14}{60})\ V_{max}$$

$$V_{max} = 60 \text{ kmph}$$

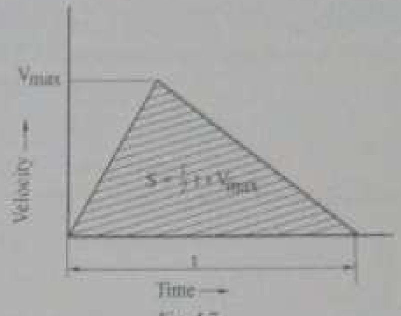


Fig. 4.7

Example 4.4.

A car travelling at 40 kmph sights a distant signal at 150 m. and comes uniformly to rest at the signal. It remains at rest for 20 s. As allowed by the signal, it uniformly accelerate and attain 40 kmph in 250 m. Calculate the time lost due to signal.

Solution

$$u = 40\text{kmph} = \frac{40 \times 5}{18} = 11.11\text{m/s}$$

$$s = 150\text{m}$$

$$v = 0$$

$$v^2 = u^2 + 2as$$

$$0 = 11.11^2 + 2 \times a \times 150$$

$$a = -0.41 \text{ m/s}^2$$

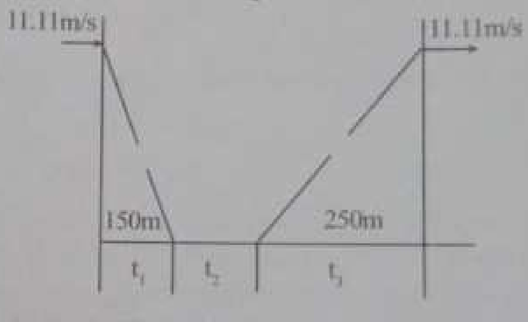


Fig. 4.8

Let t_1 be the time taken by the car to comes to rest,

$$v = u + at$$

$$0 = 11.11 + (-0.41) \times t_1$$

$$t_1 = 27 \text{ s}$$

Let t_2 be the time during which the car remains at rest

$$t_2 = 20\text{s (given)}$$

To calculate the time taken to attain 11.11 m/s in 250 m.

$$v^2 = u^2 + 2as$$

$$11.11^2 = 0^2 + 2 \times a \times 250$$

$$a = 0.247 \text{ m/s}^2$$

Let t_3 be the time taken by the car to attain the speed of 11.11m/s.

$$v = u + at$$

$$11.11 = 0 + 0.247 \times t_3$$

$$t_3 = 45s$$

Total time of travel $= t_1 + t_2 + t_3 = 27 + 20 + 45 = 92$ s

Time required to cover a distance of (150+250) = 400 m with a uniform velocity of 11.11 m/s,

$$T = \frac{400}{11.11} = 36s$$

$$\therefore \text{time lost} = 92 - 36$$

$$= 56$$

From velocity - time diagram,

$$150 = \frac{1}{2} \times t_1 \times 11.11$$

$$t_1 = \frac{300}{11.11} = 27s$$

$$t_2 = 20s \text{ (given)}$$

$$250 = \frac{1}{2} t_3 \times 11.11$$

$$t_3 = \frac{250}{11.11} = 45 \text{ s}$$

Total time of travel $= t_1 + t_2 + t_3 = 27 + 20 + 45 = 92$ s

Therefore time lost due to signal $= (t_1 + t_2 + t_3) - T = 92 - 36 = 56$ s

Motion of particle with variable acceleration.

The equations, $V = u + at$, $V^2 = u^2 + 2as$ and $s = ut + \frac{1}{2} a t^2$ are applicable only when the particle moves with uniform acceleration. When a particle is acted upon by a force which varies with time, the acceleration also varies with time. The displacement, velocity and acceleration are functions of time. Differentiating the expression for displacement with respect to time, we will get the expression for velocity and differentiating the expression for velocity with respect to time we will get the expression for acceleration. When, S = f(t),

$$V = \frac{dS}{dt} = \frac{d}{dt} f(t) \text{ and } a = \frac{dV}{dt} = \frac{d}{dt}\frac{d}{dt} f(t) = \frac{d^2}{dt^2} f(t).$$

When the expression for acceleration is given as a function of time, then by integrating the expression for acceleration we will get the expression for velocity and integrating the expression for velocity we will get the expression for displacement.

$$\text{Acceleration, } a = \frac{dV}{dt}$$

$$dV = a\,dt$$

$$V = \int a\,dt$$

$$\text{and } V = \frac{dS}{dt}$$

$$\text{That is } dS = V\,dt$$

$$S = \int V\,dt$$

When the expression for acceleration is given as a function of velocity or displacement, use the expression for acceleration, $a = V\frac{dV}{dS}$

Example 4.5.

The motion of a particle along a straight line is defined as $S = 25t + 5t^2 - 2t^3$, where S is in metres and t is in seconds. Find, (i) the velocity and acceleration at the start; (ii) the time the particle reaches maximum velocity and (iii) the maximum velocity of the particle.

Solution:

$$S = 25\,t + 5\,t^2 - 2\,t^3$$

$$\text{Velocity, } V = 25 + 10t - 6\,t^2 \text{ and}$$

$$\text{Acceleration, } a = 10 - 12\,t$$

(i) At $t = 0$

$$\text{Velocity, } V = 25 + 0 - 0 = 25 \text{ m/s}$$

$$\text{Acceleration, } a = 10 - 0 = 10 \text{ m/s}^2$$

(ii) At the maximum velocity $\frac{dV}{dt} = 0$, ie; $a = 0$

$$0 = 10 - 12\,t$$
$$t = 0.83\ s.$$

(iii) The maximum velocity is at $t = 0.83$ s.

Therefore $V_{max} = 25 + 10 \times 0.83 - 6 \times 0.83^2 = 29.17$ m/s

Example 4.6

The displacement of a particle is given by $S = t^3 - 3t^2 + 2t + 5$. Find the time at which the acceleration is zero and the time at which the velocity is 2m/s.

Solution:

$$S = t^3 - 3t^2 + 2t + 5$$
$$V = 3t^2 - 6t + 2$$
$$a = 6t - 6$$

Time at which acceleration is zero,

$$0 = 6t - 6$$
$$t = 1\ s$$

Time at which velocity is 2m/s.

$$V = 3t^2 - 6t + 2$$
$$2 = 3t^2 - 6t + 2$$
$$3t^2 - 6t + 2 - 2 = 0$$
$$3t = 6$$
$$t = 2\ s$$

Example 4.7

A point is moving in a straight line with acceleration given by $a = 15t - 20$. It passes through a reference point at $t = 0$ and another point 30m away after an interval of 5 seconds. Calculate the displacement, velocity and acceleration of the point after a further interval of 5 seconds.

Solution:

$a = 15t - 20$; at $t = 0$, $S = 0$, at $t = 5$, $S = 30$ m

$$a = \frac{dV}{dt} = 15t - 20$$

$$V = \int (15t - 20)\, dt$$

$$= \frac{15\,t^2}{2} - 20\,t + c_1$$

$$V = \frac{dS}{dt} = 7.5t^2 - 20t + c_1$$

$$S = \int (7.5t^2 - 20t + c_1)\, dt$$

$$= \frac{7.5\,t^3}{3} - \frac{20\,t^2}{2} + c_1 t + c_2$$

$$S = 2.5\, t^3 - 10\, t^2 + c_1\, t + c_2$$

At $t = 0$, $S = 0$,

$$0 = 0 - 0 + 0 + c_2$$

Therefore $c_2 = 0$

At $t = 5$, $S = 30$ m.

$$30 = 2.5 \times 5^3 - 10 \times 5^2 + c_1 \times 5 + 0$$

$$c_1 = -6.5$$

Displacement, velocity and acceleration at the end of 10s.

$$S = 2.5\, t^3 - 10\, t^2 - 6.5\, t$$

$$= 2.5 \times 10^3 - 10 \times 10^2 - 6.5 \times 10$$

Displacement $S = 1435$ m.

Velocity $V = 7.5\, t^2 - 20\, t - 6.5$.

$$V = 7.5 \times 10^2 - 20 \times 10 - 6.5$$

$$= 543.5 \text{ m/s}$$

Acceleration $a = 15t - 20$

$$a = 15 \times 10 - 20$$

$$= 130 \text{ m/s}^2$$

4.4. Kinetics

Kenetics is the study of the relation existing between the forces acting on a body, the mass of the body and the motion of the body. It is used to predict the motion caused by given forces or to determine the forces required to produce a given motion. There are three general approaches to the solution of problems in kinetics.

1. Direct application of Newton's second law.

2. Use of work-energy principles and

3. Solution by impulse and momentum.

- Each approach has its special characteristics and advantages.

4.5. Equations of motion

Equation of motion in kinetics relates force, mass and acceleration of a body. According to Newton's second law the rate of change of momentum is directly proportional to the impressed force and the motion takes place in the direction in which the force acts. The above statement leads to statement that force is directly proportional to the product of mass and acceleration. The unit of force is so selected that the constant of proportionality reduces to unity. Thus the Newton's law reduces to the statement, force = mass × acceleration. $F = m \times a$. When a system of forces act on a body, the above statement can be stated as, resultant force is equal to the product of mass and acceleration in the direction of the resultant force. Resultant force or net force = mass × acceleration.

Example 4.8

A block weighing 1000 N rest on a horizontal plane. Find the magnitude of the force required to give the block an acceleration of 2.5 m/s² to the right. The coefficient of kinetic friction between the block and the plane is 0.25

$$W = 1000\text{ N},\ a = 2.5\text{ m/s}^2,\ \mu = 0.25$$

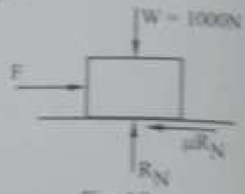


Fig. 4.9

Since there is no motion in the vertical direction,

Net force in the vertical direction = 0

$$R_N - W = 0$$

$$R_N = W = 1000\text{ N}$$

Net force in the horizontal direction = mass × acceleration

$$F - \mu R_N = m \times a$$

$$F - 0.25 \times 1000 = \frac{1000}{9.81} \times 2.5$$

$$F = 504.84\text{ N}$$

Example 4.9.

A body of mass 50 kg. slides down a rough inclined plane whose inclination to the horizontal is 30°. If the coefficient of friction between the plane and the body is 0.4, determine the acceleration of the body.

Solution.

$$m = 50\ kg \quad \alpha = 30^\circ \quad \mu = 0.4.$$

Net force along the inclined plane = mass × acceleration along the inclined plane.

$$mg \sin \alpha - \mu R_N = m \times a. \quad \text{———(i)}$$

Since there is no motion, normal to the inclined plane, net force perpendicular to the inclined plane is zero.

$$R_N - mg \cos \alpha = 0$$

$R_N = mg \cos \alpha$, substituting this value of R_N in equation (i),

$$mg \sin \alpha - \mu\ mg \cos \alpha = m \times a.$$

$$a = g \sin \alpha - \mu g \cos \alpha$$

$$= 9.81 \sin 30 - 0.4 \times 9.81 \times \cos 30$$

$$a = 1.51\ m/s^2$$

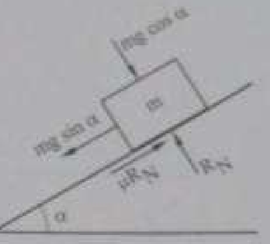


Fig. 4.10

Example 4.10.

Two blocks A and B are held stationary 10 m apart on a 20° incline as shown in Fig. 4.11. The coefficient of friction between the plane and block A is 0.3 while it is 0.2 between the plane and block B. If the blocks are released simultaneously, calculate the time taken and the distance travelled by each block before they are at the verge of collision.

Solution.

Consider the motion of block A, Net force = mass × acceleration.

$$m_A g \sin \theta - \mu R_{NA} = m_A \times a_A$$

$$m_A g \sin \theta - \mu m_A g \cos \theta = m_A \times a_A$$

$$250 \sin 20 - 0.3\ R_{NA} = \frac{250}{9.81} \times a_A$$

$$250 \sin 20 - 0.3 \times 250 \times \cos 20 = \frac{250}{9.81} \times a_A$$

$$a_A = 0.59\ m/s^2$$

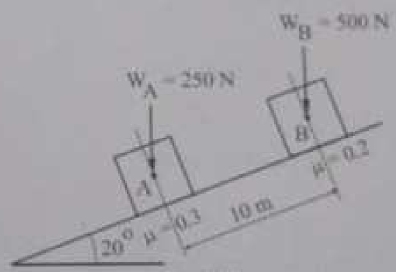


Fig. 4.11

Consider the motion of block B.

$$\text{Net force} = \text{mass} \times \text{acceleration}$$

$$m_B g \sin \theta - \mu R_{NB} = m_B \times a_B$$

$$m_B g \sin\theta - \mu m_B g \cos\theta = m_B \times a_B$$

$$500 \sin 20 - 0.2 \times R_{NB} = \frac{500}{9.81} \times a_B$$

$$a_B = 1.51 \ \text{m/s}^2$$

Let x be the distance travelled by block A in t seconds, then the distance travelled by block B in the same t second will be (10+x)

$$S_A = x$$

$$= u_A t + \frac{1}{2} a_A t^2$$

$$x = 0 + \frac{1}{2} \times 0.59 \times t^2$$

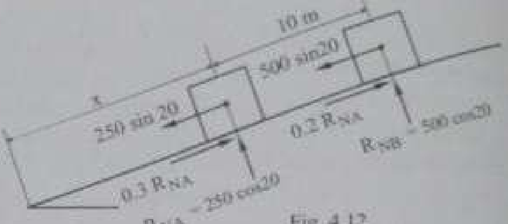


Fig. 4.12

$$10 + x = 0 + \frac{1}{2} \times a_B t^2$$

$$= \frac{1}{2} \times 1.51 \times t^2$$

$$10 + x - x = \frac{1}{2} \times 1.51 \times t^2 - \frac{1}{2} \times 0.59 \times t^2$$

$$10 = 0.46 \ t^2$$

$$t = 4.66 \ \text{s.}$$

$$x = \frac{1}{2} \times 0.59 \times 4.66^2$$

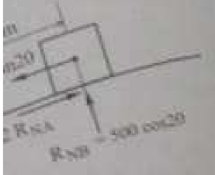
$$= \frac{1}{2} \times 0.59 \times 4.66^2$$

$$= 6.41 \ \text{m.}$$

4.6. Motion of connected bodies.

Consider two bodies connected by a light inextensible string passing over a smooth pulley. Since the pulley is smooth, the tension in the string on both sides of the pulley will be the same. The body of greater mass moves downwards and the other mass moves upwards. Considering the motion of each body separately and applying Newton's law of motion, $F = m \times a$, the acceleration of the body and the tension in the string can be determined.

nce travelled by block B



4.12

er a smooth pulley.
pulley will be the
s moves upwards.
's law of motion,
e determined.

Force in the direction of motion should be taken as positive and force opposite to the direction of motion should be taken as negative.

Example 4.11

A mass of 60 kg lies on a smooth horizontal table. It is connected to a fine string passing over a smooth guide pulley on the edge of the table to a mass 50 kg. hanging freely. Find the tension in the string and the acceleration of the system.

Solution.

$m_1 = 60$ kg $m_2 = 50$ kg.

Let T be the tension in the string.

Consider the horizontal motion of mass m_1

Net force = mass × acceleration

$T = m_1 \times a_1 = m_1 \times a$

$T = 60 \times a$ ---------(i)

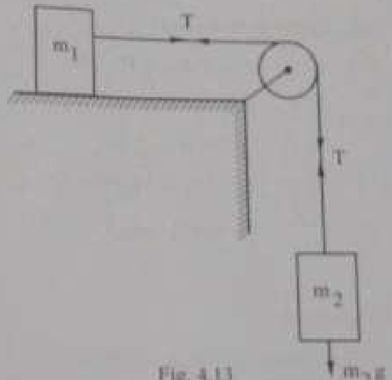


Fig. 4.13

Consider the vertical motion of mass m_2.

Net force = mass × acceleration.

$m_2g - T = m_2 \times a_2 = m_2\,a$

$50 \times 9.81 - T = 50 \times a$ ---------(ii)

Substituting for T from equation (i),

$50 \times 9.81 - 60 \times a = 50\,a$

$110\,a = 50 \times 9.81$

Acceleration, $a = 4.46\ m/s^2$

Tension, $T = 60 \times a = 60 \times 4.46$

$T = 267.6$ N

Example 4.12

Two blocks are joined by an inextensible string as shown in Fig 4.14. If the system is released from rest, determine the velocity of block after it has moved 2 m. Assume the coefficient of friction between the block and the plane is 0.25. The pulley is weightless and frictionless.

Solution

$m_A = 200$ kg, $m_B = 300$ kg, $\mu = 0.25$

Let T be the tension in the string, since $x_A = x_B$

Consider the motion of block A,

Net force = mass × acceleration.

$T - \mu R_N = m_A \times a.$

$T - 0.25 \times 200 \times 9.81 = 200 \times a$ ----------(i)

Consider vertical motion of block B

Net force = mass × acceleraton

$m_B\, g - T = m_B \times a$

$300 \times 9.81 - T = 300 \times a$ -----(ii)

Adding equations (i) and (ii)

$300 \times 9.81 - 0.25 \times 200 \times 9.81 = 500\ a$

$a = 4.905\ m/s^2$

$V^2 = u^2 + 2\,as$

$V^2 = 0 + 2 \times 4.905 \times 2$

$V = 4.43\ m/s$

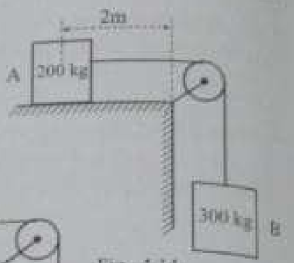


Fig. 4.14

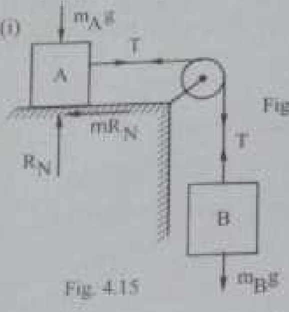


Fig. 4.15

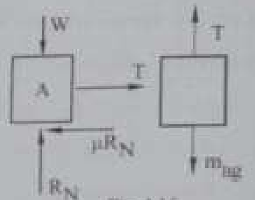


Fig 4.16

Example 4.13

The system of bodies shown in Fig. 4.17 starts from rest. Determine the acceleration of the body B and the tension in the string supporting body A.

Solution.

For a given vertical displacement of body A, the displacement of B along the inclined plane will be half of that of A.

$$x_B = \frac{1}{2}\, x_A$$

$$a_A = 2\, a_B$$

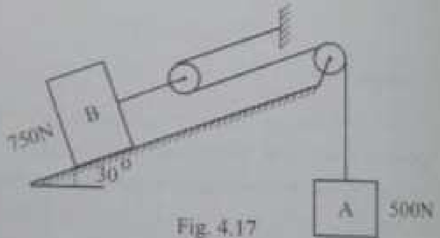


Fig. 4.17

Let T be the tension in the string.

Consider the downward motion of body A,

Net force = mass × acceleration.

$$m_A g - T = m_A \times a_A$$

$$500 - T = \frac{500}{9.81} \times a_A = \frac{500}{9.81} \times 2\, a_B$$

$$500 - T = \frac{1000}{9.81} a_B \text{ ------(i)}$$

Consider the motion of body B, up the inclined plane.

Net force = mass × acceleration.

$$2T - m_B g \sin\theta = m_B \times a_B$$

$$2\,T - 750 \times \sin 30 = \frac{750}{9.81} \times a_B$$

$$T - 187.5 = \frac{375}{9.81} \times a_B \text{ ----- (ii)}$$

Adding equations (i) and (ii)

$$312.5 = \frac{1375}{9.81} \times a_B$$

$$a_B = 2.23 \text{ m/s}^2$$

From eqn (ii)

$$T - 187.5 = \frac{375}{9.81} \times 2.23.$$

$$T = 272.74 \text{ N.}$$

Example 4.14.

Determine the tension in the string and acceleration of the two bodies of mass 300 kg and 100 kg connected by a string and frictionless and weightless pulley as shown in Fig 4.18.

Solution.

The downward displacement of 300 kg mass will be only half of the upward displacement of 100 kg mass.

$$\text{The acceleration, } a_A = \frac{1}{2}\, a_B$$

Consider the downward motion of body A

Net force = mass × acceleration

$$m_A g - 2\,T = m_A \times a_A$$

$300 \times 9.81 - 2T = 300 \times a_A$

$150 \times 9.81 - T = 150\ a_A$ ----------(i)

Consider the upward motion of body B

Net force = mass × acceleration

$T - m_B g = m_B\ a_B$

$= m_B \times 2\ a_A$

$T - 100 \times 9.81 = 100 \times 2 \times a_A$

$T - 100 \times 9.81 = 200\ a_A$ ----------(ii)

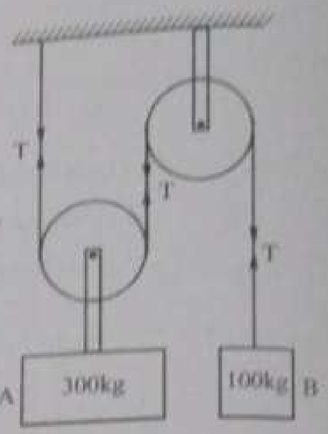


Fig. 4.18

From equations (i) and (ii)

$50 \times 9.81 = 350\ a_A$

$a_A = 1.4\ m/s^2$

$a_B = 2.8\ m/s^2$

Example 4.15.

Two smooth inclined planes whose inclinations with horizontal are 30^0 and 20^0 are placed back to back. Two bodies of mass 10 kg and 5 kg are placed on them and are connected by a string as shown in Fig. 4.19. Calculate the tension in the string and the acceleration of the bodies

Solution.

The downward displacement of body A will be equal to the upward placement of body B, along the inclined planes.

$a_A = a_B = a$

Consider the motion of A.

Net force $= m_A \times a$

$m_A g \sin \theta - T = m_A a_A$

$10 \times 9.81 \times 0.5 - T = 10 \times a$

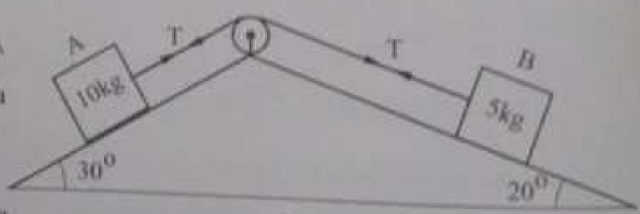


Fig. 4.19

Consider the motion of body B

$T - m_B\ g \sin 20 = m_B \times a$

$T - 5 \times 9.81 \times 0.34 = 5 \times a$ ---------(ii)

From equations (i) and (ii)

$$32.37 = 15\,a$$

$$a = 2.16\ m/s^2$$

Example 4.16

A weight, 4500 N is supported in a vertical plane by a string and pulleys arranged as shown in Fig 4.20. If the free end A of the string is pulled vertically downward with constant acceleration of 1.8 m/s², find the tension in the string.

Solution:

When the acceleration of end A is 1.8 m/s², the upward acceleration of weight W will be 0.9 m/s². Let T be the tension in the string.

Net force = mass × acceleration

$$2T - W = \frac{W}{g} \times 0.9$$

$$2T = W + \frac{W}{g} \times 0.9$$

$$= 4500\left[1 + \frac{0.9}{9.81}\right]$$

$$= 4912.84\ N$$

Tension in the string, T = 2456.42 N

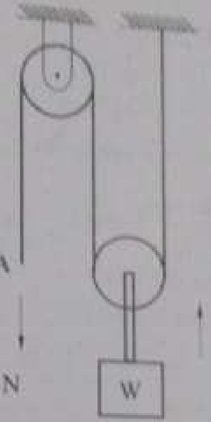


Fig. 4.20

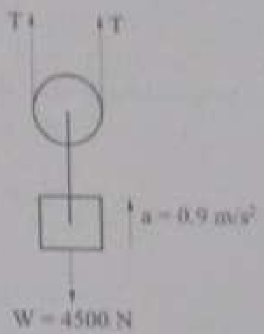


Fig. 4.21

Example 4.17.

Weights W and 2W are supported in a vertical plane by a string and pulleys arranged as shown in Fig 4.22. Find the magnitude of an additional weight Q applied to the weight W which will have a downward acceleration 0.981 m/s².

Solution:

When the downward acceleration of weight W is 0.981 m/s², the corresponding upward acceleration of weight 2W will be 0.4905 m/s².

Net force = mass × acceleration

$$2T - 2W = \frac{2W}{9.81} \times 0.4905$$

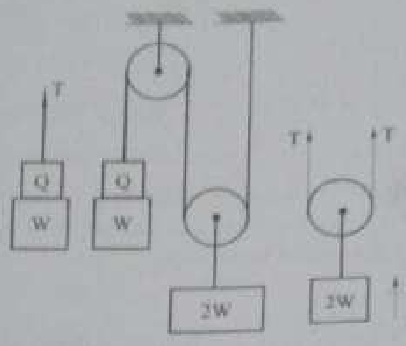


Fig. 4.22

$$T - W = 0.05\ W$$

$$T = 1.05\ W \ldots\ldots (i)$$

Downward acceleration of weight W is 0.981 m/s^2.

Net force = mass × acceleration

$$W + Q - T = \frac{W + Q}{g} \times 0.981 \ldots\ldots (ii)$$

$$= 0.1\ (W + Q)$$

$$Q - 0.1Q = 0.1W + 1.05W - W$$

$$0.9Q = 0.1W - W + 1.05\ W$$

$$= 0.15W$$

$$Q = 0.167W$$

Example 4.18 | KTU Jan. 2016 |

Two equal weights W are connected by a string passing over a frictionless pulley. A small weight w is attached to one side, as shown in Fig. 4.23, causing that the weight to fall. Determine the acceleration of the system assuming that the weights starts from rest.

Solution

Consider the upward motion of weight W,

Net force = mass × acceleration

$$T - W = \frac{W}{g} \times a \ldots\ldots (i)$$

Consider the downward motion of weight (W+w)

Net force = mass × acceleration

$$W + w - T = \frac{W + w}{g} \times a \quad \text{...... (ii)}$$

Adding equations (i) and (ii)

$$T - W + W + w - T = \frac{W}{g} \times a + \left(\frac{W + w}{g}\right) \times a$$

$$w = (W + W + w) \times \frac{a}{g}$$

$$= (2W + w) \times \frac{a}{g}$$

$$a = \frac{wg}{2W + w}$$

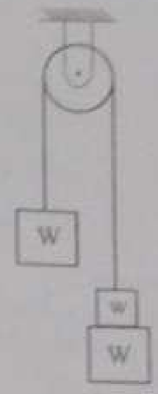


Fig. 4.23

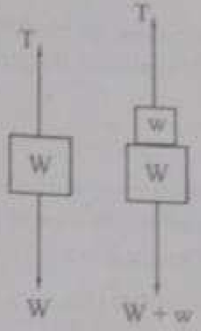


Fig. 4.24

4.7. D'Alembert's principle.

D'Alembert's principle is an application of Newton's second law to a moving body. A problem in dynamics can be converted into an equivalent problem in static using D'Alembert's principle. Newton's law of motion F = ma can be written as F – ma = 0. The term (– ma) is called inertia force and is denoted by F_I. According to Newton's first law of motion, a body continues to be in a state of rest or of uniform motion along a straight line unless acted by an external unbalanced force. Thus every body has a tendency to continue in its state of rest or of uniform motion. This tendency is called inertia. The magnitude of inertia force is equal to the product of the mass and acceleration and it acts in a direction opposite to the direction of acceleration. F = ma can be written as F – ma = 0, or F + (– ma) = 0, $F + F_I = 0$. The statement of the above equation is known as D'Alembert's principle which states that the resultant of a system of force acting on a body in motion is in dynamic equilibrium with the inertia force.

Example 4.19.

A force of 300 N acts on a body of mass 150 kg. Calculate the acceleration of the body using D'Alembert's principle.

Solution.

$$F = 300\ N,\ m = 150\ kg$$

$$F + (-ma) = 0$$

$$300 + (-150 \times a) = 0$$

$$300 = 150\ a$$

$$a = \frac{300}{150} = 2\ m/s^2$$

Example 4.20.

A system of weights connected by strings passing over pulleys A and B is shown in Fig.4.25. Find the acceleration of the three weights P, Q and R. Using D'Alembert's principle.

Solution.

Let the downward acceleration of weight P be a. Then the upward acceleration of pulley B is a. Let the downward acceleration of weight Q be a_1 with respect to pulley B. Then upward acceleration of weight R is a_1

Absolute acceleration of weight Q is $a_1 - a$

Absolute acceleration of weight R is $a_1 + a$

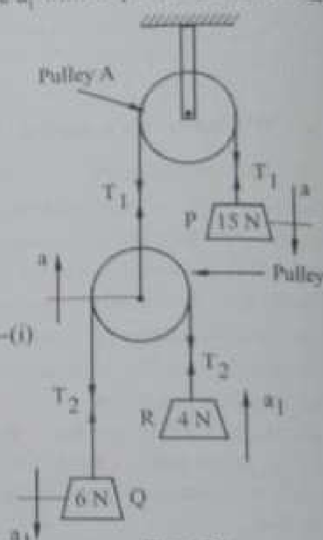


Fig. 4.25

Consider the downward motion of weight P.

$$F + (-ma) = 0$$

$$15 - T_1 - \frac{15}{9.81} \times a = 0$$

$$T_1 = 15 - 1.53\ a \text{ ----(i)}$$

Consider the downward motion of weight Q.

$$F + (-ma) = 0$$

$$6 - T_2 - \frac{6}{9.81} \times (a_1 - a) = 0$$

$$T_2 = 6 - 0.61(a_1 - a) \text{ ---(ii)}$$

Consider the upward motion of weight R

$$F + (-ma) = 0$$

$$T_2 - 4 - \frac{4}{9.81}(a_1 + a) = 0$$

$$T_2 = 4 + 0.41\,(a_1 + a) \text{ ----- (iii)}$$

Equating eqns (ii) and (iii)

$$T_2 = 6 - 0.61\,(a_1 - a) = 4 + 0.41\,(a_1 + a)$$

$$1.02\,a_1 - 0.2a = 2$$

$$a_1 - 0.196\,a = 1.96 \text{ ----- (iv)}$$

Consider the motion of the weightless pulley B.

$$F + (-\,ma) = 0$$

$$F = 0$$

$$2\,T_2 - T_1 = 0$$

$$T_1 = 2\,T_2$$

Form eqn (i)

$$2\,T_2 = 15 - 1.53\,a$$

$$T_2 = 7.5 - 0.765\,a \text{ ----------- (v)}$$

Equating eqns (iii) and (v)

$$6 - 0.61\,(a_1 - a) = 7.5 - 0.765\,a$$

$$-\,a_1 + 2.25\,a = 2.46 \text{ ----------- (vi)}$$

Adding equations (iv) and (vi),

$$2.054\,a = 4.42$$

$$a = 2.15 \text{ m/s}^2$$

From eqn (iv)

$$a_1 - 0.196\,a = 1.96$$

$$a_1 = 1.96 + 0.196 \times 2.15$$

$$= 2.38 \text{ m/s}^2$$

Acceleration of $P = a = 2.15 \text{ m/s}^2$

Acceleration of $Q = a_1 - a = 2.38 - 2.15$

$$= 0.23 \text{ m/s}^2$$

Acceleration of weight $R = a_1 + a = 2.38 + 2.15$

$$= 4.53 \text{ m/s}^2$$

Motion of lift.

Consider the motion of a lift with acceleration in the downward direction. Let W be the weight of a man and R be the reaction of force applied by the man on the floor of the lift.

Acceleration is downwards and hence the inertia force acts in the upward direction.

For dynamic equilibrium,

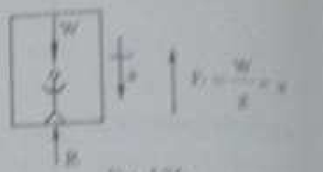


Fig. 4.26

$$R + F_i - W = 0$$

$$R = W - F_i$$

$$= W - \frac{W}{g} a$$

$$R = W \left(1 - \frac{a}{g}\right)$$

When the acceleration is upwards, the direction of inertia force is downwards.

For dynamic equilibrium,

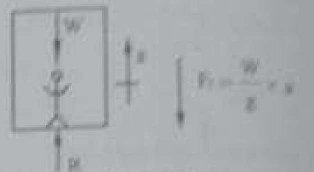


Fig. 4.27

$$R - W - F_i = 0$$

$$R = W + F_i$$

$$= W + \frac{W}{g} a$$

$$R = W \left[1 + \frac{a}{g}\right]$$

When a lift moves with uniform velocity, the acceleration of lift is zero. A man standing on the floor of the lift exerts force equal to his own weight on the lift. When lift moves with acceleration in downward direction the man exerts less force on the floor of the lift and when the lift moves up with acceleration the man exerts more force on the floor of the lift.

When the lift accelerates, the direction of inertia force is opposite to the direction of acceleration and when the lift decelerates, the direction of inertia force is in the same direction of deceleration.

Example 4.21 [KTU Jan 2016, June 2016, May 2019]

A lift has an upward acceleration of 1.2 m/s². What force will a man weighing 750 N exerts on the floor of the lift ? What force would he exert if the lift had an acceleration of 1.2 m/s² downwards. What upward acceleration would cause his weight to exert a force of 900 N on the floor.

Solution.

Case (i). When the lift moves upward.

$$a = 1.2 \text{ m/s}^2$$

$$W = 750 \text{ N}$$

$$R = W\,[1 + \frac{a}{g}] = 750\left[1 + \frac{1.2}{9.81}\right]$$

$$= 841.74 \text{ N}$$

Case (ii) When the lift moves downward.

$$a = 1.2 \text{ m/s}^2$$

$$W = 750 \text{ N}$$

$$R = W\,[1 - \frac{a}{g}]$$

$$= 750\left[1 - \frac{1.2}{9.81}\right]$$

$$= 658.26 \text{ N}$$

Case (ii) the required acceleration for R = 900 N

When lift moves up

$$W = 750 \text{ N}$$

$$R = 900 \text{ N}$$

$$R = W\,[1 + \frac{a}{g}]$$

$$900 = 750\left[1 + \frac{a}{9.8}\right]$$

$$a = 1.96 \text{ m/s}^2$$

Example 4.22.

An elevator of total weight 5000 N starts to move upwards with a constant acceleration of 1 m/s^2. Find the force in the cable during the acceleration motion. Also find the force at the floor of the elevator under the feet of a man weighing 600 N when the elevator moves up with a uniform retardation of 1 m/s^2.

Solution.

$$W = 5000 \text{ N}.$$

$$a = 1 \text{ m/s}^2$$

Case (i) Elevator moves upwards with acceleration.

In this case the accelerating force is upwards and inertia force is downwards.

For dynamic equilibrium of elevator,

$$T - W - F_i = 0$$

$$T = W + \frac{W}{g} a$$

$$T = W \left[1 + \frac{a}{g}\right]$$

$$= 5000\left[1 + \frac{1}{9.81}\right]$$

$$= 5509.68 \text{ N}$$

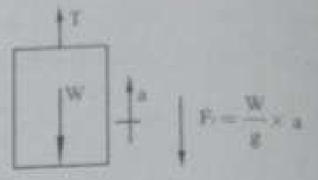


Fig. 4.28

Case (ii) When the elevator moves up with uniform deceleration, the inertia force is upwards.

Consider the dynamic equilibrium of the man

Weight of man, $W = 600$ N.

$$R + F_i - W = 0$$

$$R = W - F_i$$

$$= W - \frac{W}{g} a$$

$$R = W \left[1 - \frac{a}{g}\right]$$

$$= 600 \left[1 - \frac{1}{9.81}\right]$$

$$= 538.84 \text{ N}$$

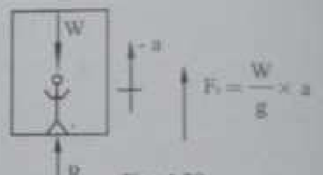


Fig. 4.29

Example 4.23.

An elevator has an upward acceleration of 1 m/s^2. What pressure will be transmitted to the floor of the elevator by a man weighing 600 N travelling in the elevator ? What pressure will be transmitted if the elevator has a downward acceleration of 2 m/s^2 ?

Solution.

Upward motion.

Let R be the reaction of pressure exerted by the man on the floor and W be the weight of man.

For dynamic equilibrium,

$$R - W - F_i = 0$$

$$R = W + F_i$$

$$= W + \frac{W}{g} \times a$$

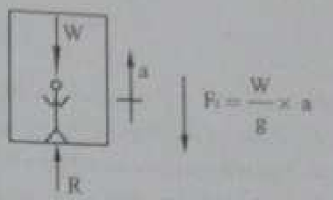


Fig. 4. 30

$$= W\left[1 + \frac{a}{g}\right]$$

$$= 600\left[1 + \frac{1}{9.81}\right]$$

$$R = 661.16 \text{ N.}$$

Downward motion.

For dynamic equilibrium,

$$R + F_i - W = 0$$

$$R = W - F_i$$

$$= W - \frac{W}{g} \times a$$

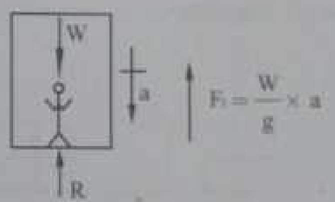


Fig. 4.31

$$= W\left[1 - \frac{a}{g}\right]$$

$$= 600\left[1 - \frac{2}{9.81}\right]$$

$$R = 477.68 \text{ N}$$

4.8. Work energy equation in rectilinear translation

In mechanics, work is said to be done whenever the point of application of a force is moved along the line of action of that force or along the line of action of the component of the that force. The work done by a force F during a differential displacement dS of its point of application is F cos θ dS, where θ is the angle between F and dS. F cos θ is the force in the direction of displacement. Alternatively the work done may be interpreted as the force multiplied by the displacement component ds cos θ in the direction of the force.

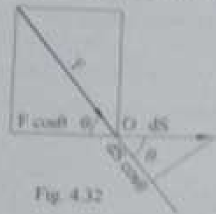


Fig. 4.32

Work done, dW = F dS cos θ. Work is positive when the force or component of force is in the direction of displacement and negative if it is in the opposite direction. Work done by force of gravity is positive when a body moves from a higher elevation to a lower elevation. Work done by force of friction is always negative because direction of frictional force is always opposite to the direction of motion of the body.

The unit of work is Newton-metre (N.m) . 1 Nm = 1 joule (J). It is the work done by a force of 1 N moving through a distance of 1 m in the direction of the force.

During a finite displacement S of the point of application of a force F, the force does an amount of work W.

$$W = \int F\cos\theta \times dS$$

Now we shall discuss different cases.

Case (i) When force F is constant and θ = 0.

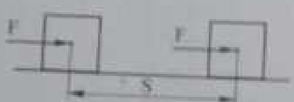


Fig. 4.33

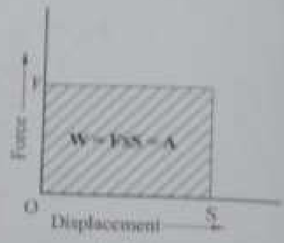


Fig. 4.34

Work done = F × S

Case (ii) When F is constant and is inclined θ° with the direction of motion.

Fig. 4.35

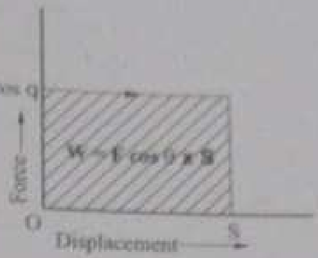


Fig. 4.36

Work done, $W = F \cos\theta \times S$

Case (iii) When Force F varies linearly and $\theta = 0$.

Work done W = average force during the displacement × displacement

$$W = \frac{(0 + F)}{2} \times S = \frac{1}{2} F \times S$$

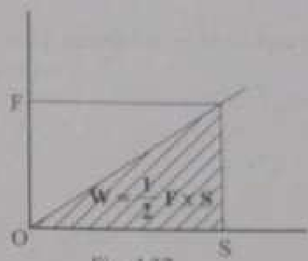


Fig. 4.37

Case (iv) When $F = f(S)$.

Work done W = area under the curve 1-2 of the force displacement diagram.

$$W = \int_1^2 F \times dS$$

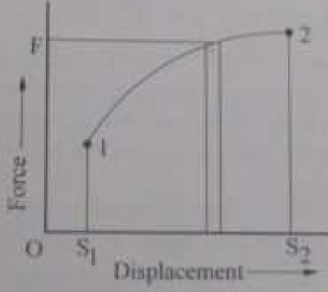


Fig. 4.38

A common example of work done by a variable force is the work of extension or compression of a spring. Consider a spring of stiffness k. The spring force is proportional to the deformation of the spring.

Spring force α x

Spring force = k x, where k is called stiffness of spring. When a spring of stiffness k is

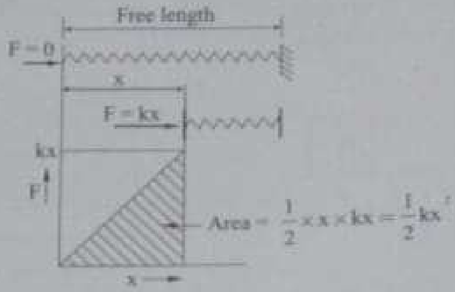


Fig. 4.39

compressed by an amount x from its free length, then the spring force varies from 0 to k x.

Work done = average force × displacement

$$= \left(\frac{0 + k\,x}{2}\right)x$$

$$W = \frac{1}{2}\,k\,x^2$$

Energy.

Energy is the capacity to do work. The unit of energy is same as that of work. The kinetic energy of a body of mass m, moving with a velocity V is $\frac{1}{2}$ m V^2. Potential energy of a body of weight W held at a height h is W h = m g h.

The work energy principle states that the work done by a system of force acting on a body during a displacement is equal to the change in kinetic energy of the body during the same displacement. Consider a body of mass m moving with a velocity u. Let S be the displacement of the body and V be the final velocity. Let F be the resultant force acting on the body in the direction of displacement.

Resultant force F = m × a , where a is the acceleration of the body.

$$a = \frac{dV}{dt} = \frac{dV}{dS}\times\frac{dS}{dt} = \frac{dV}{dS}\times V$$

$$a = V\,\frac{dV}{ds}$$

$$F = m \times a$$

$$= m \times V\,\frac{dV}{ds}$$

$$F \times dS = m\, V\, dV$$

Integrating on both sides,

$$\int_0^s F\, dS = \int_u^V m\, V\, d\, V$$

$$F \times S = m \left[\frac{V^2}{2} \right]_u^V$$

$$= \frac{1}{2}\, m\, (V^2 - u^2)$$

$$= \frac{1}{2}\, m\, V^2 - \frac{1}{2}\, mu^2$$

Work done = Change in K.E

Example 4.24.

Calculate the work done in pulling up a block weighing 20 kN for a length of 5 m on a smooth plane inclined 20^0 with the horizontal.

Solution.

When a body is kept on an inclined plane, the gravity force acting along the plane is mg sin θ

$$\text{Work done} = m\, g\, \sin\theta \times S$$

$$= 20 \times 10^3 \times \sin 20 \times 5$$

$$= 34202 \text{ J}$$

4.9. Impulse momentum equation.

Principle of impulse and momentum is derived from Newton's second law, F = ma. This principle relates force, mass, velocity and time and, is suitably used for solving problems where large forces act for a very small time.

When a large force acts over a short period of time, that force is called an impulsive force.

The impulse of a force F acting over a time interval t_1 to t_2 is defined by the integral, $\int_{t_1}^{t_2} F\, dt$

If F is the resultant force acting on a body of mass m, then from Newton's second law,

$$F = ma$$

$$a = \frac{dv}{dt}$$

$$\therefore F = m \times \frac{dv}{dt}$$

$$= \frac{d}{dt}(mv)$$

The product of mass and velocity is called momentum. ie., the resultant force acting on a body is equal to the rate of change of momentum of the body.

$$F = \frac{d}{dt}(mv)$$

$$F\,dt = d\,(mv)$$

$$\int F\,dt = \int d\,mv$$

$$= m\int dv$$

$$\int_{t_1}^{t_2} F\,dt = m\,[V]_{v_1}^{v_2}$$

$$= (mv_2 - mv_1)$$

$$F(t_2 - t_1) = mv_2 - mv_1$$

$$Ft = m(v_2 - v_1)$$

Impulse = Final momentum - Initial momentum.

Unit of impulse is Ns.

Unit of momentum is

$$kg\,\frac{m}{s} = kg\,\frac{m}{s^2} \times s = N.s$$

Example 4.25.

A body weighing 40 N drops freely from a height of 50 m and penetrates into the ground by 100 cm. Find the average resistance to penetration and the time of penetration.

Solution.

Velocity body when it just strikes the ground,

$$V = \sqrt{2gh} = \sqrt{2g \times 50}$$

$= 31.32$ m/s.

Velocity of body after penetrating 100 cm = 0.

Using work energy principle.

Change in K.E = Work done

Let P be the average resistance of penetration

$$\frac{1}{2} m (V_2^2 - V_1^2) = mgx - P \times x$$

$$\frac{1}{2} \times \frac{40}{9.81} (0^2 - 31.32^2) = 40 \times 1 - P \times 1$$

$$P = 40 + \frac{1}{2} \times \frac{40}{9.81} \times 31.32^2$$

$$= 2039.88 \text{ N}$$

Using impulse momentum equation

$$F \times t = m (V_2 - V_1)$$

F is the resultant force on the body during penetration.

$$F = mg - P$$

$$= 40 - 2039.88$$

$$= -1999.88 \text{ N}$$

$$-1999.88 \times t = \frac{40}{9.81} (0 - 31.32)$$

$$t = 0.064 \text{ s}$$

Example 4.26.

An automobile weighing 25 kN is moving at a speed of 60 kmph, when the brakes are fully applied causing all four wheels to skid. Determine the time required to stop the automobile. The coefficient of friction between the road and tyre is 0.5.

Solution.

Initial velocity, $V_1 = 60$ kmph

$$= 60 \times \frac{5}{18}$$

$$= 16.67 \text{ m/s.}$$

Final velocity, $V_2 = 0$

The average force during stopping $F = \mu R_N$

R_N = Normal reaction = Weight of automobile

$= 25 \times 10^3$ N

$F = \mu R_N$

$= 0.5 \times 25 \times 10^3 \text{ N} = 12.5 \times 10^3 \text{ N}$

Applying impulse momentum equation,

$$F \times t = m(V_2 - V_1)$$

$$-12.5 \times 10^3 \times t = \frac{25 \times 10^3}{9.81}(0 - 16.67)$$

$$t = 3.4 \text{ s}$$

4.10. Equations of kinematics in curvilinear translation

Whenever a particle moves it describes a path. When the path is a curve the particle is said to have curvilinear motion. When this curved path lies in a plane, the motion is called plane curvilinear motion. When the curve is circular, the motion is called circular motion or motion of rotation. To define the position of a particle in a plane, two coordinates, x and y are required. As the particle moves, these coordinates change with time and hence x and y are functions of time.

$$x = f_1(t) \text{ and } y = f_2(t).$$

$x = f_1(t)$ represents the rectilinear motion along the x axis of the projection P_x of the particle P moving along the curved path as shown in Fig.4.40. $y = f_2(t)$ represents the rectilinear motion along the y axis of the projection P_y of the particle P. Thus the curvilinear motion of a particle P may be considered as the resultant of the rectilinear motions of its projections P_x and P_y along the x and y axes.

The displacemnt of P in Δt seconds is the vector sum of Δx and Δy.

$$\Delta S = \Delta x + \Delta y.$$

Dividing by Δt,

$$\frac{\Delta S}{\Delta t} = \frac{\Delta x}{\Delta t} + \frac{\Delta y}{\Delta t} \text{ ------ (i)}$$

If Δt is infinitely diminised to the limit of zero, the ratio $\frac{\Delta S}{\Delta t}$ becomes the instantaneous velocity of the particle.

$$V = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \frac{dS}{dt}$$

the particle is
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The velocities along x and y axes are given by,

$$V_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \text{ and}$$

$$V_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$$

From equation (i),

$$\frac{dS}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

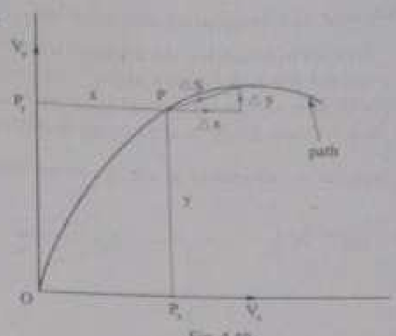
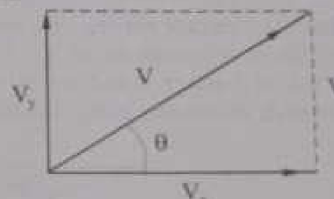


Fig. 4.40

$V = V_x + V_y$, the vector sum of V_x and V_y. Since V_x and V_y are perpendicular, the magnitude of velocity, $V = \sqrt{V_x^2 + V_y^2}$ and direction of velocity, $\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)$.

When the velocity of the particle changes by ΔV during a time interval Δt, the average acceleration during the displacement is $\frac{\Delta V}{\Delta t}$. In the limit Δt tends to zero, the average acceleration tends to insantaneous acceleration a. Thus,

$$a = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt}.$$



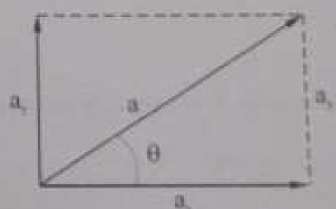
$$a_x = \lim_{\Delta t \to 0} \frac{\Delta V_x}{\Delta t} = \frac{dV_x}{dt} \text{ and}$$

$$a_y = \lim_{\Delta t \to 0} \frac{\Delta V_y}{\Delta t} = \frac{dV_y}{dt}$$

We have, $V = V_x + V_y$

Differentiating with respect to time,

$$\frac{dV}{dt} = \frac{dV_x}{dt} + \frac{dV_y}{dt}$$



$a = a_x + a_y$, the vector sum of accelerations along X and Y axes. Since a_x and a_y are perpendicular, the magnitude of acceleration $a = \sqrt{a_x^2 + a_y^2}$ and the direction of acceleration $\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right)$.

Normal and tangential acceleration

Instead of resolving the acceleration along the x and y axes, it is sometimes convenient to resolve the acceleration along normal and tangential to the path. Such components of acceleration are called normal acceleration and tangential acceleration and are denoted by a_n and a_t respectively. Since normal and tangential components are perpendicular, the magnitude of acceleration at the given instant is $\sqrt{a_n^2 + a_t^2}$ and the direction is given by

$$\theta = \tan^{-1}\left(\frac{a_n}{a_t}\right).$$

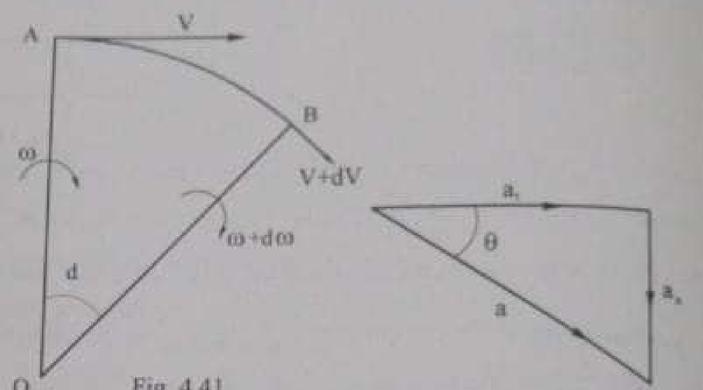
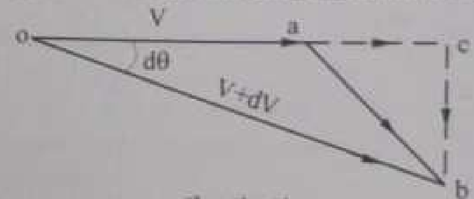


Fig. 4.41

Consider a particle moving on a curved path from A to B in dt seconds. Let the angular rotation in dt seconds be dθ. Let V and V + dV be the instantaneous velocities of the particle at A and B. ω and ω + dω be the angular velocities of particle at A and B. The change in the velocity in dt seconds is the vector difference of V and V + dV.



The change in veocity in dt seconds, $\vec{ab} = \vec{ac} + \vec{cb}$.

Dividing by dt,

$$\frac{\vec{ab}}{dt} = \frac{\vec{ac}}{dt} + \frac{\vec{cb}}{dt}$$

$\frac{\vec{ac}}{dt}$ is the tangential component of acceleration and $\frac{\vec{cb}}{dt}$ is the normal component of acceleration.

$$a_t = \frac{\vec{ac}}{dt} = \frac{\vec{oc} - \vec{oa}}{dt}$$

$$= \frac{(V + dV)\cos d\theta - V}{dt}$$

When $d\theta$ tends to zero, $\cos d\theta$ tends to 1.

$$\therefore a_t = \frac{(V + dV) - V}{dt} = \frac{dV}{dt}$$

$$= \frac{d}{dt}(r\omega)$$

$$= r\frac{d\omega}{dt}$$

$$= r\alpha$$

$$a_t = \frac{dV}{dt} = r\alpha$$

The tangential component is due to the change in the magnitude of velocity. When a particle moves with uniform velocity the tangential component of acceleration is zero.

The normal component of acceleration,

$$a_n = \frac{\vec{cb}}{dt} = \frac{(V + dV)\sin d\theta}{dt}$$

When $d\theta$ tends to zero, $\sin d\theta$ tends to $d\theta$,

$$= \frac{(V + dV)\,d\theta}{dt}$$

$$= \frac{V\,d\theta + dV\,d\theta}{dt},$$

neglecting the product of two small values dV and $d\theta$,

$$a_n = V\frac{d\theta}{dt} = V \cdot \omega = r\omega \cdot \omega$$

$$= \omega^2 r = \left(\frac{V}{r}\right)^2 \times r = \frac{V^2}{r}$$

$$a_n = V\frac{d\theta}{dt} = \frac{V^2}{r} = \omega^2 r$$

The normal component of acceleration is due to change in direction of velocity and hence when a particle moves along a straight path, the normal component of acceleration is zero.

The normal component of acceleration is always directed towards the centre of rotation and is also called centripetal component of acceleration denoted by a_c.

Example 4.27.

The motion of a particle is described by the following equations,

$$x = 2(t+1)^2 \text{ and } y = \frac{2}{(t+1)^2}$$

Show that the path travelled by the particle is rectangular hyperbola. Find the velocity and acceleration of the particle at t = 1s.

Solution:

$$\text{Given: } x = 2(t+1)^2 \text{ and } y = \frac{2}{(t+1)^2}$$

$$x \times y = 2(t+1)^2 \times \frac{2}{(t+1)^2} = 4$$

Since $x \times y$ is a constant, the path travelled by the particle is rectangular hyperbola.

$$x = 2(t+1)^2$$

$$V_x = \frac{dx}{dt} = 2 \times 2(t+1)$$

$$V_x = 4(t+1)$$

$$a_x = \frac{dV_x}{dt} = \frac{d}{dt}[4(t+1)] = 4 \text{ m/s}^2$$

$$\text{at } t = 1s, \quad V_x = 4(1+1)$$

$$= 8 \text{ m/s}$$

$$y = \frac{2}{(t+1)^2}$$

$$V_y = \frac{dy}{dt} = 2 \times (-2) \times (t+1)^{-3}$$

$$= \frac{-4}{(t+1)^3}$$

$$a_y = \frac{dV_y}{dt} = \frac{d}{dt}\frac{(-4)}{(t+1)^3} = \frac{12}{(t+1)^4}$$

at t = 1s, $$V_y = \frac{-4}{(t+1)^3} = -0.5 \text{ m/s}$$

$$a_y = \frac{12}{(1+1)^4} = 0.75 \text{ m/s}^2$$

Velocity of particle after 1s.

$$V = \sqrt{V_x^{\,2} + V_y^{\,2}}$$

$$= \sqrt{8^2 + (-0.5)^2}$$

$$= 8.02 \text{ m/s}$$

Acceleration of particle after 1s $$a = \sqrt{a_x^{\,2} + a_y^{\,2}}$$

$$= \sqrt{4^2 + 0.75^2}$$

$$= 4.07 \text{ m/s}^2$$

Example 4.28.

Prove that if the ends A and B of a bar AB of length l are constrained to move along the X and Y axes respectively, its midpoint C describes a circle of radius $\frac{l}{2}$ with centre at the origin O and any intermediate point D describes an ellipse with semi major and semi minor axes $\left(\frac{l}{2} + b\right)$ and $\left(\frac{l}{2} - b\right)$ respectively.

Solution:

At any instant when the bar AB is inclined θ° with horizontal, the coordinates of mid point C,

$$x = \frac{l}{2}\cos\theta \text{ and } y = \frac{l}{2}\sin\theta$$

Squaring and adding,

$$x^2 + y^2 = \left(\frac{l}{2}\right)^2\left[\cos^2\theta + \sin^2\theta\right]$$

$x^2 + y^2 = \left(\frac{l}{2}\right)^2$ which repersents a circle of radius $\left(\frac{l}{2}\right)$ with centre at the origin O.

The coordinates of D are,

$x_D = \left(\frac{l}{2} + b\right)\cos\theta$ and $y_D = \left(\frac{l}{2} - b\right)\sin\theta$

$\cos\theta = \frac{x_D}{\frac{l}{2} + b}$ and $\sin\theta = \frac{y_D}{\left(\frac{l}{2} - b\right)}$

Squaring and adding,

$$\frac{x_D^2}{\left(\frac{l}{2} + b\right)^2} + \frac{y_D^2}{\left(\frac{l}{2} - b\right)^2} = \cos^2\theta + \sin^2\theta$$

$$= 1$$

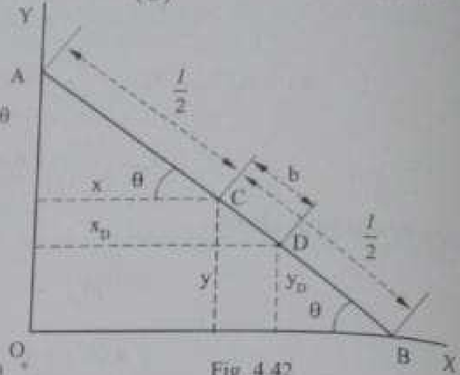


Fig. 4.42

which represents an ellipse of semimajor axis $\left(\frac{l}{2} + b\right)$ and semiminor axis $\left(\frac{l}{2} - b\right)$

Example 4.29.

A particle moves with a constant speed of 6m/s along the parabolic path, $y = kx^2$. Determine its acceleration at a point (10m, 5m) on the parabola.

Solution:

Since the particle moves with constant speed, its tangential component of acceleration is zero.

The normal component of acceleration $a_n = \frac{V^2}{\rho}$, where ρ is the radius of curvature of the path.

$$y = kx^2,$$

at A, y = 5m and x = 10 m.

$$5 = k \times 10^2$$

$$k = \frac{1}{20}$$

$$\therefore y = \frac{x^2}{20}$$

The radius of curvature ρ is given by

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

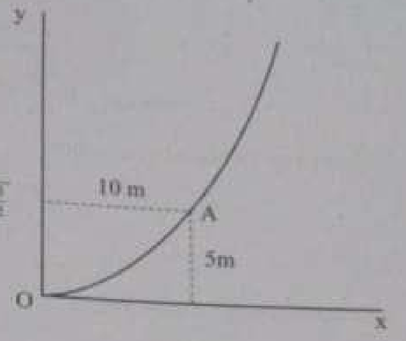


Fig. 4.43

$$y = \frac{x^2}{20}$$

$$\frac{dy}{dx} = \frac{x}{10} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{10}$$

$$\frac{1}{\rho} = \frac{\left(\dfrac{1}{10}\right)}{\left[1+\left(\dfrac{x}{10}\right)^2\right]^{\frac{3}{2}}}, \text{ at } x = 10 \text{ m.}$$

$$\frac{1}{\rho} = \frac{\dfrac{1}{10}}{(1+1)^{\frac{3}{2}}}$$

$$\rho = 28.28 \text{ m}$$

The normal component of acceleration,

$$a_n = \frac{V^2}{\rho} = \frac{6^2}{28.28} = 1.273 \text{m/s}^2$$

$$\text{Acceleration, } a = \sqrt{a_n^2 + a_t^2} = \sqrt{(1.273)^2 + 0^2} = 1.273 \text{ m/s}^2$$

Example 4.30.

A particle moves along a circular path of radius r such that the distance covered is given by $s = ct^2$, where c is a constant. Find the tangential and normal component of acceleration of the particle.

Solution:

$$s = ct^2$$

$$\text{Velocity,} \quad V = \frac{ds}{dt} = 2ct$$

Tangential component of acceleration,

$$a_t = \frac{dV}{dt} = \frac{d}{dt}(2ct)$$

$$= 2c$$

Normal component of acceleration,

$$a_n = \frac{V^2}{r} = \frac{(2ct)^2}{r}$$

$$= \frac{4c^2t^2}{r}$$

Example 4.31.

A car starts from rest on a curved road of radius 600 m and aquires by the end of the first 60 seconds of motion a speed of 24 kmph. Find the tangential and normal acceleration at the instant, t = 30 seconds.

Solution:

Since the car starts from rest, $V_1 = 0$, $\omega_1 = 0$.

$$\text{After 60s, } V_2 = 24 \text{ kmph} = 24 \times \frac{5}{18} \text{ m/s}$$

$$\text{Velocity, } V_2 = r \cdot \omega_2$$

$$\omega_2 = \frac{V_2}{r} = \frac{24 \times \frac{5}{18}}{600} = 0.011 \text{ rad/s}$$

To calculate the angular acceleration α,

$$\omega_2 = \omega_1 + \alpha t$$

$$0.011 = 0 + \alpha \times 60$$

$$\alpha = 0.00018 \text{ rad/s}^2$$

at t = 30s, angular velocity,

$$\omega = \omega_1 + \alpha t$$

$$= 0 + 0.00018 \times 30$$

$$= 0.0055 \text{ rad/s}$$

Tangential component of acceleration,

$$a_t = r\alpha$$

$$= 600 \times 0.00018$$

$$= 0.108 \text{ m/s}^2$$

Normal component of acceleration,

$$a_n = \omega^2 r$$

$$= 0.0055^2 \times 600$$

$$= 0.018 \text{ m/s}^2$$

Example 4.32.

A car starts from rest on a curved road of 250 m radius and accelerates at a constant tangential acceleration of 0.6 m/s². Determine the distance and the time for which that car will travel before the magnitude of the total acceleration attained by it becomes 0.75 m/s².

Solution:

At point B, the total acceleration, $a = 0.75 \text{ m/s}^2$ and tangential acceleration is 0.6 m/s^2.

$$a = \sqrt{a_n^2 + a_t^2}$$

$$a^2 = a_n^2 + a_t^2$$

$$a_n^2 = a^2 - a_t^2$$

$$= 0.75^2 - 0.6^2 = 0.2025$$

$$a_n^2 = a^2 - a_t^2 = 0.75^2 - 0.6^2 = 0.2025$$

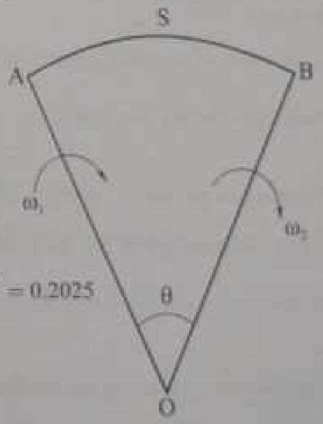


Fig. 4.44

Normal acceleration at B, $a_n = \omega_2^2 r = \sqrt{0.2025} = 0.45$

Angular velocity at B, $\omega_2 = \sqrt{\dfrac{a_n}{r}} = \sqrt{\dfrac{0.45}{250}} = 0.0424 \text{ rad/s}$

The tangential acceleration at B $a_t = 0.6\ m/s^2 = r\alpha$

$$\text{Angular acceleration } \alpha = \frac{0.6}{250} = 0.0024\ rad/s^2$$

$$\omega_2 = \omega_1 + \alpha t$$

$$0.0424 = 0 + 0.0024 \times t$$

$$\text{Time of travel, } t = \frac{0.0424}{0.0024} = 17.67s$$

Distance travelled, $S = AB = r\theta$

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$$

$$= 0 + \frac{1}{2} \times 0.0024 \times 17.67^2 = 0.375\ rad$$

$\therefore$ The distance travelled $S = r\theta = 250 \times 0.375$

$$= 93.75m$$

Example 4.33.

A car enters a curved portion of a road, in the form of a quarter of a circle, of radius 100 m at 18 kmph and leaves at 36 kmph. If the car is travelling with a constant tangential acceleration, find the magnitude and direction of acceleration when the car (i) enters and (ii) leaves the curved portion of road.

Solution:

Linear velocity at entrance is 18 kmph = 5 m/s

Angular velocity at entrance is, $\omega_1 = \frac{V}{r} = \frac{5}{100} = 0.05$ rad/s

Normal component of acceleration at entrance, $a_n = \omega_1^2 r = 0.05^2 \times 100 = 0.25\ m/s^2$

Linear velocity when the car leaves the curved portion is 36 kmph = 10 m/s

Normal component of acceleration when the car leaves, $a_n = \omega_2^2 r = 0.1^2 \times 100 = 1\ m/s^2$

Angular velocity when the car leaves the road, $\omega_2 = \frac{V}{r} = \frac{10}{100} = 0.1$ rad/s

$\omega_1 = 0.05 rad/s, \omega_2 = 0.1 rad/s$ and $\theta = \frac{\pi}{4}$

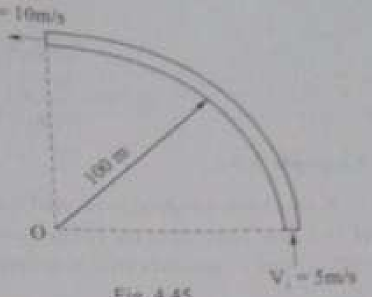


Fig. 4.45

$$\omega_2^2 = \omega_1^2 + 2\alpha\theta$$

$$0.1^2 = 0.05^2 + 2 \times \alpha \times \frac{\pi}{4}$$

∴ Angular acceleration $\alpha = 0.0024 \text{ rad/s}^2$

Tangential acceleration, $a_t = r\alpha$

$$= 100 \times 0.0024 \text{ rad/s}^2$$

$$= 0.24 \text{ m/s}^2$$

Acceleration of the car when it enters the road is the vector sum of normal and tangential acceleration.

$$a = \sqrt{a_t^2 + a_n^2}$$

$$= \sqrt{0.24^2 + 0.25^2}$$

$$= 0.35 \text{ m/s}^2$$

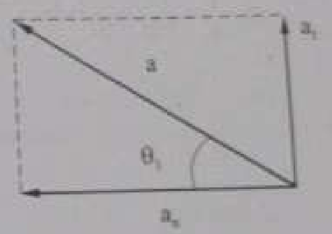


Fig. 4.46

The direction of acceleration with horizontal,

$$\tan\theta_1 = \frac{a_t}{a_n}$$

$$\theta_1 = \tan^{-1}\frac{a_t}{a_n} = \tan^{-1}\frac{0.24}{0.25}$$

$$= 43.83^0$$

When the car leaves the road, the constant tangential acceleration,

$a_t = 0.24 \text{ m/s}^2$ and normal acceleration is 1 m/s^2

∴ Acceleration of the car when it leaves the road,

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{0.24^2 + 1^2}$$

$$= 1.03 \text{ m/s}^2$$

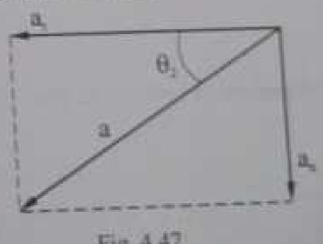


Fig. 4.47

The direction of acceleration with horizontal is given by

$$\tan\theta_2 = \frac{a_t}{a_n} = \frac{0.24}{1} = 0.24$$

$$\theta_2 = 13.5^\circ$$

Example 4.34.

A car enters a curved portion of the road of radius 200 m, travelling at a constant speed of 36 kmph. Determine the components of velocity and acceleration of the car in the X and Y directions 15 seconds after it has entered the curved portion of the road.

Solution:

Since the car moves with uniform speed, change in velocity dV = 0, Therefore after 15 seconds the velocity of car is 36 kmph (10m/s) itself. This velocity is tangential to the curved path as shown in Fig 4.48. Let α be the inclination of V_t with horizontal. $\alpha = 90 - \theta$, where θ is the angular displacement after 15 seconds.

$$\theta = \omega \times t$$

$$\omega = \frac{V}{R} = \frac{10}{200} = \frac{1}{20} \text{ rad/s}$$

$$\therefore \theta = \frac{1}{20} \times 15 = 0.75 \text{ rad}$$

$$= 42.97^\circ$$

$$\alpha = 90 - \theta$$

$$= 90 - 42.97$$

$$= 47.03^\circ$$

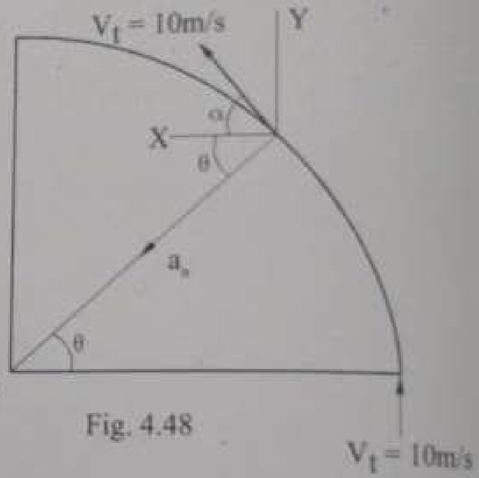


Fig. 4.48

Component of velocity in X direction is $V_t \cos\alpha$

$$= 10 \times \cos 47.03$$

$$= 6.82 \text{ m/s}$$

Component of velocity in the Y direction is $V_t \sin\alpha$

$$= 10 \times \sin 47.03$$

$$= 7.32 \text{ m/s}$$

Since $dV = 0$, the tangential component of acceleration $a_t = \frac{dV}{dt} = 0$.

The normal component of acceleration $a_n = \omega^2 r = \frac{V^2}{r} = \frac{10^2}{200} = 0.5\ m/s^2$

It is directed towards the centre of rotation.

$$\text{Total acceleration, } a = \sqrt{a_t^2 + a_n^2}$$

$$= a_n$$

$$= 0.5\ m/s^2$$

It is inclined θ° with X axis.

Component of acceleration in X direction,

$$= a\cos\theta = 0.5 \times \cos 42.97$$

$$= 0.37\ m/s^2$$

Component of acceleration in Y direction,

$$= a\sin\theta = 0.5\sin 42.97$$

$$= 0.34\ m/s^2$$

Example 4.35.

A particle moves along a curve $x = 0.64\,y^2$. Its law of motion is $x = 4t^2$, where x and y are in meters and t in second. At $t = 3s$, calculate,

(i) the displacement of the particle from the origin.

(ii) velocity of the particle.

(iii) acceleration of the particle.

Solution:

The equation of motion is $x = 4t^2$.

The equation of curve is $x = 0.64\,y^2$.

$$\text{at } t = 3s,\ x = 4 \times 3^2$$

$$= 36\ m$$

$$36 = 0.64\,y^2$$

$$y = 7.5\ m$$

at $t = 3$ s, $x = 36$ m and $y = 7.5$ m

The displacement of particle from the origin is $\sqrt{x^2+y^2}$

$$= 36.77 \text{ m}$$

Given that $x = 4t^2$

Velocity in the x direction,

$$V_x = \frac{dx}{dt} = 8t$$

Again, $x = 4t^2$ and

$$x = 0.64\, y^2$$

$$4t^2 = 0.64\, y^2$$

$$y^2 = 6.25\, t^2$$

$$y = 2.5\, t$$

Velocity in y direction, $V_y = \frac{dy}{dt} = 2.5 \text{ m/s}$

at $t = 3s$, $V_x = 8t = 8 \times 3 = 24 \text{ m/s}$

$$V_y = 2.5 \text{ m/s}$$

$\therefore$ Velocity, $V = \sqrt{V_x^2 + V_y^2}$

$$= \sqrt{24^2 + 2.5^2}$$

$$= 24.13 \text{ m/s}$$

Acceleration in the X direction, $a_x = \frac{dV_x}{dt} = \frac{d(8t)}{dt}$

$$= 8 \text{ m/s}^2$$

Acceleration in the Y direction, $a_y = \frac{dV_y}{dt} = \frac{d(2.5)}{dt}$

$$= 0$$

$\therefore$ Acceleration, $a = \sqrt{a_x^2 + a_y^2} = \sqrt{8^2 + 0^2}$

$$= 8 \text{ m/s}^2$$

Example 4.36

The rectangular components of a particle moving in a curved path is given by $V_x = 2t - 3$ and $V_y = 3t^2 - 12t + 12$. The co-ordinates of a point on the path at an instant, $t = 0$ are (4, –8). Establish the equation of path.

Solution:

Given that, $V_x = 2t - 3$

$V_y = 3t^2 - 12t + 12$

$$V_x = \frac{dx}{dt} = (2t - 3)$$

$$dx = (2t - 3)\,dt$$

Integrating, $$x = \frac{2t^2}{2} - 3t + c_1$$

$$= t^2 - 3t + c_1$$

at $t = 0$, $x = 4m$

$$4 = 0 - 0 + c_1$$

$$c_1 = 4$$

$\therefore$ $$x = t^2 - 3t + 4 \quad \text{.............. (i)}$$

$$V_y = \frac{dy}{dt} = 3t^2 - 12t + 12$$

$$dy = (3t^2 - 12t + 12)\,dt$$

Integrating, $$y = \frac{3t^3}{3} - \frac{12t^2}{2} + 12t + c_2$$

at $t = 0$, $y = -8$.

$$-8 = 0 - 0 + 0 + c_2$$

$$c_2 = -8$$

$\therefore$ $$y = t^3 - 6t^2 + 12\,t - 8$$

$$= (t - 2)^3$$

$$t - 2 = y^{\frac{1}{3}}$$

$$t = y^{\frac{1}{3}} + 2$$

Substituting this value of t in the expression for x,

$$x = t^2 - 3t + 4$$

$$x = \left[y^{\frac{1}{3}} + 2\right]^2 - 3\left[y^{\frac{1}{3}} + 2\right] + 4$$

$$= y^{\frac{2}{3}} + 4 + 4y^{\frac{1}{3}} - 3y^{\frac{1}{3}} - 6 + 4$$

$$x = y^{\frac{2}{3}} + y^{\frac{1}{3}} + 2$$

4.11. Motion of projectile

A particle which is projected into space at an angle to the horizontal is called a projectile. The path traced by the projectile is called trajectory. The velocity with which the projectile is projected into space is called velocity of projection and, the angle with the horizontal, at which the projectile is projected is called angle of projection. The time during for which the

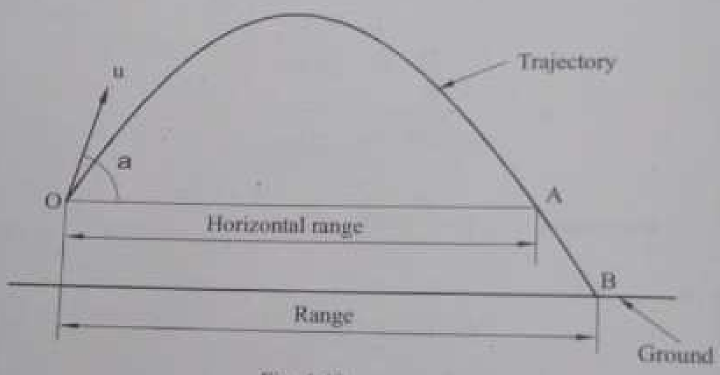


Fig. 4.49

projectile is in motion is called time of flight. It is the interval of time since the projectile is projected and hits the ground. Range is the horizontal distance between the point of projection and the point where the projectile strikes the ground. Horizontal range is the horizontal distance between the point of projection and the point where the horizontal line through the point of projection meets the trajectory. The point from which the projectile is projected into

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space is called point of projection.

O is the point of projection.

u is the velocity of projection.

α is the angle of projection.

OA is the horizontal range.

B is the point at which the projectile strikes the ground.

Horizontal distance between O and B is the range of projectile.

Motion of a particle projected vertically into space.

Let u be the velocity of projection.

$$\alpha = 90^0$$

The velocity of the particle at a certain height h can be obtained using the relation,

$$V^2 = u^2 - 2\,gh$$

When, $h = h_{max}$, $V = 0$

$$0 = u^2 - 2\,gh_{max}$$

$$h_{max} = \frac{u^2}{2g}$$

Fig. 4.50

The time to attain maximum height can be obtained using the relation,

$$V = u - g\,t$$

$$0 = u - g\,t_1$$

Time to attain maximum height $t_1 = \frac{u}{g}$

Therefore, time of flight $T = 2\,t_1 = \frac{2u}{g}$

Since $\alpha = 90^0$, the particle will come back to the point of projection after T seconds. Hence the range of particle is zero.

Motion of a particle thrown horizontally into space.

Consider a particle thrown horizontally from a point A, h, above the ground as shown in Fig 4.51. At any instant, the particle is subjected to

i) horizontal motion with constant velocity u.

ii) vertical downward motion with initial velocity zero and acceleration due to gravity g.

Consider the vertical motion.

Using the general expression,

$$h = u\,t + \frac{1}{2} g t^2$$

$$h = 0 + \frac{1}{2} g t^2$$

Time of flight $T = t = \sqrt{\left[\frac{2h}{g}\right]}$

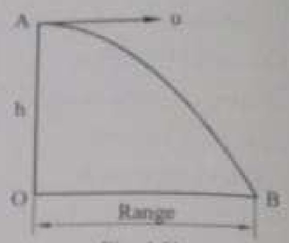

Fig. 4.51

During this period the particle moves horizontally with uniformly velocity u m/s.

Therefore, range = u × T

$$= u \sqrt{\left[\frac{2h}{g}\right]}$$

Inclined projection on a level ground.

Consider the motion of a projectile projected from point O with a velocity of projection u and angle of projection α. The projectile has motion in vertical as well as horizontal directions.

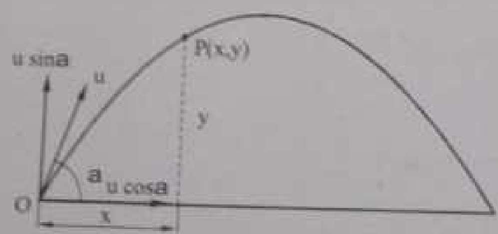

Fig. 4.52

Since there is no force (neglecting air resistance) in the horizontal direction, horizontal component of velocity remains constant throughout the flight. Horizontal component of velocity, u cos α is constant. The vertical component of velocity decreases due to gravity force. The height of projectile from the ground h at any instant of time t sec is given by

$$h = (u \sin \alpha)\, t - \frac{1}{2}\, g\, t^2$$

Where u sin α is the initial velocity in the vertical direction. Let P (x, y) be the position of the particle after t second. x is the horizontal distance travelled in t second and y is the vertical distance travelled in t second. Since the horizontal component of velocity is constant, the horizontal distance travelled in t second,

$$x = (u \cos \alpha) \times t$$

$$t = \frac{x}{u \cos \alpha} \quad \text{- - - -(i)}$$

Using the relation $h = (u \sin \alpha)\, t - \frac{1}{2} g t^2$

$$y = (u \sin \alpha)\, t - \frac{1}{2} g t^2$$

Substituting for t from equation (i),

$$y = u \sin \alpha \times \frac{x}{u \cos \alpha} - \frac{1}{2} g \left(\frac{x}{u \cos \alpha}\right)^2$$

$$y = x \tan \alpha - \frac{1}{2} \frac{g x^2}{u^2 \cos^2 \alpha}$$

This is an equation of a parabola. Hence the trajectory is a parabola.

Expression for maximum height.

Initial velocity in the vertical direction is u sin α. At maximum height the vertical velocity is zero. ie, at $h = h_{max}$, $V = 0$

Using the relation for vertical motion,

$$V^2 = u^2 - 2 g h$$

$$0 = (u \sin \alpha)^2 - 2 g h_{max}$$

$$\therefore h_{max} = \frac{u^2 \sin^2 \alpha}{2g}$$

Expression for time to attain maximum height.

Using the relation,

$$V = u - g t$$

$$0 = (u \sin \alpha) - g t$$

$$t = \frac{u \sin \alpha}{g}$$

$$\text{Time of flight, } T = 2t = 2 \times \frac{u \sin \alpha}{g}$$

$$T = \frac{2u \sin \alpha}{g}$$

Expression for horizontal range, R.

Horizontal range R is the horizontal distance travelled in 2 t seconds where t is the time taken to attain the maximum height. Since the horizontal component of velocity is constant and equal to u cos α,

$$R = (u \cos \alpha) \times 2t$$

$$= 2u \cos \alpha \times \frac{u \sin \alpha}{g}$$

$$= \frac{u^2 \, 2 \sin \alpha \cos \alpha}{g}$$

$$R = \frac{u^2}{g} \sin 2\alpha$$

Example 4.37

A pilot flying his bomber at a height of 1000 m with uniform horizontal velocity of 30 m/s wants to strike a target on the ground. At what distance from the target, he should release the bomb?

Solution: Consider the vertical motion of the bomb. Initial vertical velocity = 0.

$$h = ut + \frac{1}{2} g t^2$$

$$1000 = 0 + \frac{1}{2} g \times t^2$$

$$t = \sqrt{\frac{2000}{g}} = 14.29 \text{ s}$$

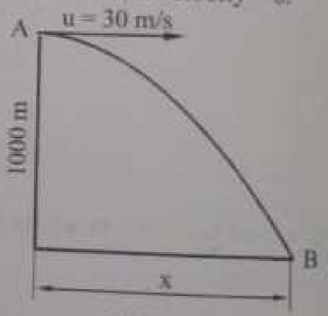

Fig. 4.53

Since the horizontal velocity remains constant, the horizontal distance moved in 14.29 seconds is velocity × time.

$$x = 30 \times 14.29 = 428.7 \text{ m.}$$

Example 4.38.

An aeroplane is flying at a height of 200 m with a horizontal velocity of 70 m/s. A shot is fired from a gun from the ground when the aeroplane is exactly above the gun. What should be the minimum initial velocity of the shot and the angle of elevation in order to hit the aeroplane.

Solution:

Let u_A be the velocity of plane and u_S be the velocity of projection of shot at an angle of projection α.

When the shot hits the aeroplane, the horizontal distance travelled by the shot and plane will be the same.

$$\therefore u_A \times t = u_S \cos\alpha \times t$$

$$u_A = u_S \cos\alpha \text{ ---------(i)}$$

The vertical component of u_S should be such that, it should go upto a height of 200 m.

Using the relation, $V^2 = u^2 - 2gh;$

$$0 = (u_S \sin\alpha)^2 - 2 \times g \times 200$$

$$u_S \sin\alpha = 62.64 \text{ ---------(ii)}$$

$$\frac{u_S \sin\alpha}{u_S \cos\alpha} = \frac{62.64}{70} = 0.895$$

$$\tan\alpha = 0.895$$

$$\alpha = 41.82^0$$

From equation (i)

$$u_A = u_S \cos\alpha$$

$$70 = u_S \cos 41.82$$

$$\therefore u_S = 93.93 \text{ m/s}$$

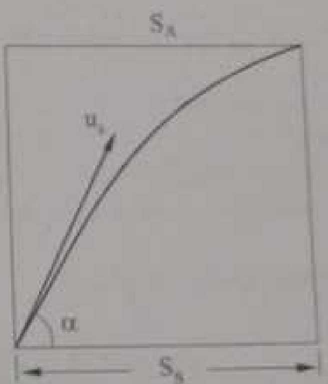

Fig. 4.54

Example 4.39

A particle is projected with a given velocity u at an angle of elevation α from the origin. It passes through two points (15, 8) and (40, 9) on its path. Find the greatest height reached by the particle and its range.

Solution:

The equation of trajectory is,

$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha} \quad x_1 = 15,\ y_1 = 8$$

$$8 = 15\tan\alpha - \frac{1}{2}\times\frac{9.81\times15^2}{u^2\cos^2\alpha}$$

$$15\tan\alpha - 8 = \frac{1}{2}\times\frac{9.81\times15^2}{u^2\cos^2\alpha} \quad \text{.......................... (i)}$$

$x_2 = 40,\ y_2 = 9$

$$9 = 40\tan\alpha - \frac{1}{2}\times\frac{9.81\times40^2}{u^2\cos^2\alpha}$$

$$40\tan\alpha - 9 = \frac{1}{2}\times\frac{9.81\times40^2}{u^2\cos^2\alpha} \quad \text{.......................... (ii)}$$

$$\frac{40\tan\alpha - 9}{15\tan\alpha - 8} = \frac{40^2}{15^2}$$

$$40\tan\alpha - 9 = 7.11(15\tan\alpha - 8)$$

$$= 106.65\tan\alpha - 56.88$$

$$66.65\tan\alpha = 47.88$$

$$\tan\alpha = 0.72$$

$$\alpha = 35.75^\circ$$

From equation (i),

$$15\tan 35.75 - 8 = \frac{1}{2}\times\frac{9.81\times15^2}{u^2\cdot\cos^2 35.75}$$

$$u = 24.47 \text{ m/s}$$

Greatest height, $h_{max} = \dfrac{u^2\sin^2\alpha}{2g}$

$$= \frac{24.47^2\sin^2 35.75}{2\times9.81}$$

$$= 10.42 \text{ m}$$

The range $R = \frac{u^2 \sin 2\alpha}{g}$

$$= \frac{24.47^2 \times \sin(2 \times 35.75)}{9.81}$$

$$= 57.88 \text{ m}$$

4.12. Equation of kinetics in curvilinear motion.

The differential equations of curvilinear motion, $F_x = m \times a_x$ and $F_y = m \times a_y$ can be written in the form $F_x - m a_x = 0$ and $F_y - m a_y = 0$. These equations have the same form of equations of static equilibrium, $\Sigma F_x = 0$, $\Sigma F_y = 0$. $-m a_x$ and $-m a_y$ are the inertia forces along the x and y axes, F_{I_x} and F_{I_y}.

$$F_x + (-m a_x) = 0$$

$$F_x + F_{I_x} = 0$$

$$F_y + (-m a_y) = 0 \quad \text{or}$$

$$F_y + F_{I_y} = 0$$

F_x and F_y are the components of resultant force acting on the moving body along x and y directions and F_{I_x} and F_{I_y} are the inertia force along x and y directions. Thus the net external force acting on the body along with the inertia force keeps the body in dynamic equilibrium. This apparent transformation of a problem in dynamics to one in statics is D'Alembert's principle.

Example 4.40

A body of weight W is suspended in a vertical plane by two strings as shown in Fig. 4.55. Determine the tension T in the inclined string, OA,

(i) an instant before the horizontal string is cut, and,

(ii) an instant just after the string is cut.

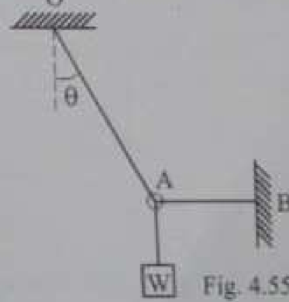

Fig. 4.55

Solution:

(i) An instant before the horizontal string is cut, the system is in static equilibrium.

Resolving the forces acting at A, vertically

$$T \cos \theta = W$$

$$T = \frac{W}{\cos\theta}$$

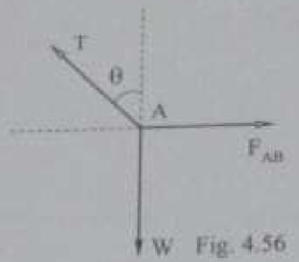

Fig. 4.56

(ii) When the string OA is cut, the body starts moving along a curved path of radius OA. The body is in dynamic equilibrium with the introduction of inertia force $\frac{W}{g} \times a_t$ as shown in Fig 4.57. Resolving the forces in the direction of tension T,

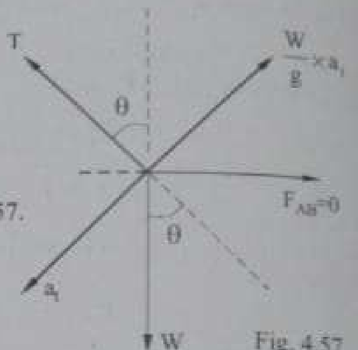

Fig. 4.57

$$T - W\cos\theta = 0$$

$$T = W\cos\theta$$

Example 4.41

A ball of weight 50N is supported in a vertical plane as shown in Fig 4.58. Find the compressive force in the bar BC just after the string AB is cut. Neglect the weight of the bar BC.

Solution:

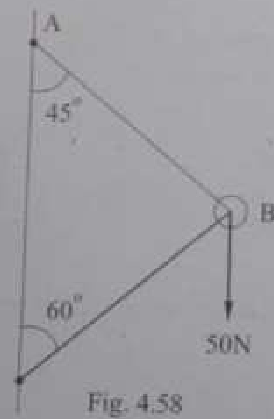

Fig. 4.58

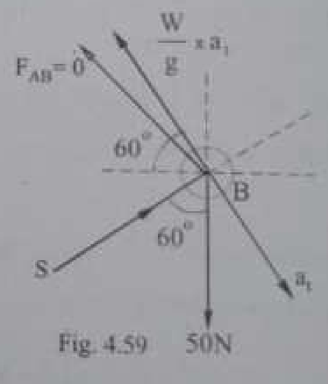

Fig. 4.59

Considering the dynamic equilibrium of the ball with the forces acting on it as shown in Fig. 4.59

Resolving the forces in the direction of the compressive force S,

$$S - 50 \cos 60 = 0$$

$$S = 50 \cos 60$$

$$= 25 \text{ N}$$

4.13. Moment of momentum

The product of mass and velocity is called momentum. It is a vector in the same direction of velocity. Consider a particle moving along the curved path AB as shown in Fig 4.60. The velocity of the particle is tangential to the path. Let V be the velocity of particle at P and V_x and V_y be the components of this velocity in the x and y directions. Let the co-ordinates of P be (x,y). The momentum of the particle is m× V and moment of this momentum about the origin O is the product of momentum and the perpendicular distance OC.

Momentum at P is m× V. The rectangular components of this momentum are mV_x and mV_y, along x and y directions. Since the moment of resultant with respect to O is equal to the algebric sum of the moments of its components with respect to the same point O. The moment of momentum,

$$H_0 = m \times V \times OC$$

$$= mV_x \times y - mV_y \times x$$

$$= m\left[V_x y - V_y x\right]$$

Clockwise moment is taken as positive and counter clockwise moment is taken as negative.

The resultant force F acting at P can be resolved in to recangular components F_x and F_y and the moment of resultant force F about O is equal to the algebric sum of moments of F_x and F_y about O. The moment of force about O,

$$M_0 = F \times OC = F_x \times y - F_y \times x$$

The equation of motion of the particle at P is given by

$F_x = m \times a_x$ and $F_y = m \times a_y$. a_x and a_y are the components of acceleration a at P along x and y directions. Multiplying F_x by y and F_y by x,

$$F_x \times y = m\, a_x \times y \text{ and}$$

$$F_y \times x = m\, a_y \times x$$

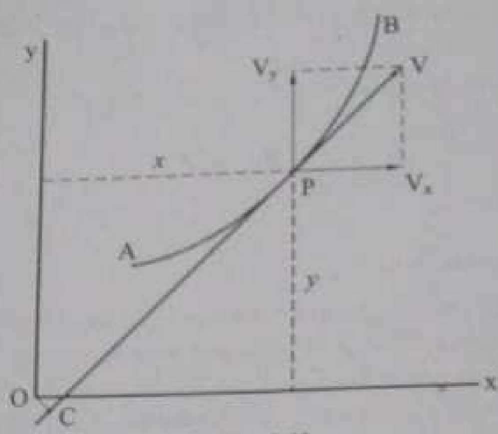

Fig. 4.60

$$F_x \times y - F_y \times x = m\,(a_x\, y - a_y\, x)$$

$$= \frac{d}{dt}\left[m\left(V_x y - V_y x\right)\right]$$

$$M_o = \frac{d}{dt} H_o$$

This equation states that, the moment of the resultant force acting on a particle with respect to any point in its plane of motion is equal to the rate of change of moment of momentum of the particle with respect to the same point.

Example 4.42

A particle of mass 1 kg is moving with a velocity of 5m/s as shown in Fig 4.61 The coordinates of the particle are (3,2). Find the angular momentum about the origin O.

Solution:

mass m = 1 kg

$V = 5$ m/s

$V_x = 5 \cos 60$

$V_y = 5 \sin 60$

$x = 3$ m

$y = 2$ m

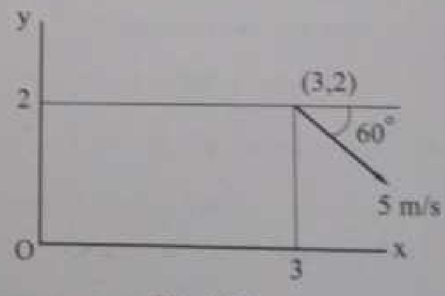

Fig. 4.61

Angular momentum about O,

$$H_O = m[V_x \times y + V_y \times x]$$

$$= 1[5\cos 60 \times 2 + 5\sin 60 \times 3] = 18 \text{ Nms}$$

Example 4.43

The motion of a particle of mass m in the x - y plane is defined by the equations x = a cos ωt and y = b sin ωt. Where a, b and ω are constants. Calculate the moment of momentum of the particle with respect to the origin.

Solution:

Moment of momentum $= m(V_x \times y - V_y \times x)$

$$x = a\cos\omega t$$

$$V_x = -a\omega\sin\omega t$$

$$y = b\sin\omega t$$

$$V_y = b\,\omega\cos\omega t$$

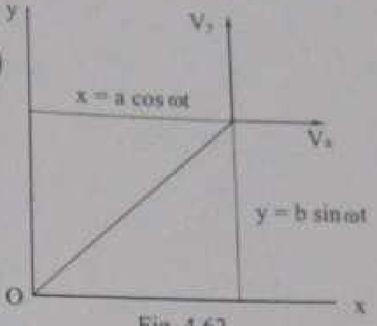

Fig. 4.62

Moment of momentum $= m[(-a\omega\sin\omega t)\times b\sin\omega t - b\omega\cos\omega t \times a\cos\omega t]$

$$= -m\,a\,b\omega\left[\sin^2\omega t + \cos^2\omega t\right]$$

$$= -m\,ab\,\omega$$

4.14 Work energy equation in curvilinear motion

When the resultant force and resultant acceleration are resolved along the tangent and normal to the curved path, the differential equations of curvilinear motion at a given instant are,

$$F_t = m \times a_t \text{ and } F_n = m \times a_n$$

$$F_t = m \times a_t$$

$$= m \times \frac{dV}{dt}, \text{ multiplying both sides by } \frac{ds}{dt}$$

$$F_t \times \frac{ds}{dt} = m\,\frac{dV}{dt} \times \frac{ds}{dt}$$

th respect
entum of

4.61 The
O.

$$F_t \times \frac{ds}{dt} = m \times V \times \frac{dV}{dt}$$

$$F_t \times ds = m \times V \times dV$$

Integrating the above expression,

$$\int_1^2 F_t ds = \int_1^2 mV \times dV$$

$$= m \times \left[\frac{V^2}{2}\right]_1^2$$

$$= \frac{1}{2} m \left(V_2^2 - V_1^2\right)$$

$$= \frac{1}{2} mV_2^2 - \frac{1}{2} mV_1^2$$

$\int_1^2 F_t\, ds$ is the work done by the resultant force acting on the body in between positions 1 and 2. $\left(\frac{1}{2} mV_2^2 - \frac{1}{2} mV_1^2\right)$ is the change in kinetic energy between the two positions 2 and 1.

Thus the work energy principle for curvilinear motion states that the change in kinetic energy of a body between any two positions is equal to the work done by the tangential components of the forces acting upon it during the motion between these two positions.

Example 4.44

A simple pendulum is released from rest at A with the string horizontal and swings downward. Express the velocity of the bob as a function of the angle θ. Also obtain the expression for angular velocity of bob when, the string is in the vertical position.

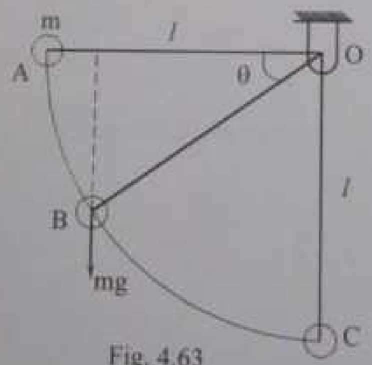

Fig. 4.63

Solution:

Work done when the pendulum swings from OA to OB is mg × vertical distance between A and B

$$= mg \times l \sin\theta$$

$$\text{Change in kinetic energy} = \frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2$$

$$= \frac{1}{2}mV_B^2$$

Equating the work done and change in kinetic energy

$$mg\,l\sin\theta = \frac{1}{2}mV_B^2$$

$$V_B = \sqrt{2gl\sin\theta}$$

When the string is vertical, $\theta = 90^\circ$

$$\therefore V_c = \sqrt{2gl\sin 90}$$

$$= \sqrt{2gl}$$

$$\text{Angular velocity } \omega = \frac{V}{r} = \frac{\sqrt{2gl}}{l}$$

$$= \sqrt{\frac{2g}{l}}$$

Example 4.45

A particle of weight W starts from rest at A and slides under the influence of gravity along a smooth track AB in a vertical plane. Find the velocity of the particle at B.

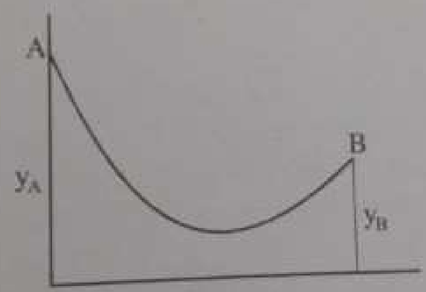

Fig. 4.64

Solution:

Work done when the particle moves from A to B is W × Vertical distance between A and B.

$$\text{Workdone} = W\left(y_A - y_B\right)$$

Change in kinetic energy between A and B is,

$$\frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2 = \frac{1}{2}mV_B^2$$

Equating the workdone and the change in kinetic energy,

$$W\left(y_A - y_B\right) = \frac{1}{2}mV_B^2$$

$$= \frac{1}{2}\frac{W}{g}V_B^2$$

$$V_B = \sqrt{2g\,W\left(y_A - y_B\right)}$$